The Mathematics Teacher

APRIL 1958

Miniature number systems
BURTON W. JONES

Symbols and functions
IRVIN H. BRUNE

Facts, fancy, fabrications, and figures
M. H. AHRENDT

Can we continue to ignore these warnings?
JOINT COMMITTEE ON ATOMIC ENERGY

The changing concept of trigonometry as a school subject
HERMAN ROSENBERG

The Mathematics Teacher *is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.*

Editor and Chairman of the Editorial Board

H. VAN ENGEN, *University of Wisconsin, Madison, Wisconsin*

Assistant Editor

I. H. BRUNE, *Iowa State Teachers College, Cedar Falls, Iowa*

Editorial Board

JACKSON B. ADKINS, *Phillips Exeter Academy, Exeter, New Hampshire*
MILDRED KEIFFER, *Cincinnati Public Schools, Cincinnati, Ohio*
Z. L. LOFLIN, *Southwestern Louisiana Institute, Lafayette, Louisiana*
PHILIP PEAK, *Indiana University, Bloomington, Indiana*
ERNEST RANUCCI, *State Teachers College, Union, New Jersey*
M. F. ROSSKOPF, *Teachers College, Columbia University, New York 27, New York*

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N. W., Washington 6, D. C.

Officers for 1957–58 and year term expires

President

HOWARD F. FEHR, *Teachers College, Columbia University, New York 27, New York, 1958*

Past-President

MARIE S. WILCOX, *Thomas Carr Howe High School, Indianapolis 7, Indiana, 1958*

Vice-Presidents

LAURA K. EADS, *New York City Public Schools, New York, New York, 1958*
DONOVAN A. JOHNSON, *University of Minnesota, Minneapolis 14, Minnesota, 1958*
ALICE M. HACH, *Racine Public Schools, Racine, Wisconsin, 1960*
ROBERT E. PINGRY, *University of Illinois, Urbana, Illinois, 1960*

Executive Secretary

M. H. AHRENDT, *1201 Sixteenth Street, N. W., Washington 6, D. C.*

Board of Directors

JACKSON B. ADKINS, *Phillips Exeter Academy, Exeter, New Hampshire, 1958*
IDA MAY BERNHARD, *Texas Education Agency, Austin 11, Texas, 1958*
HENRY SWAIN, *New Trier Township High School, Winnetka, Illinois, 1958*
PHILLIP S. JONES, *University of Michigan, Ann Arbor, Michigan, 1959*
H. VERNON PRICE, *University High School, Iowa City, Iowa, 1959*
PHILIP PEAK, *Indiana University, Bloomington, Indiana, 1959*
CLIFFORD BELL, *University of California, Los Angeles 24, California, 1960*
ROBERT E. K. ROURKE, *Kent School, Kent, Connecticut, 1960*
ANNIE JOHN WILLIAMS, *Julian S. Carr Junior High School, Durham, North Carolina, 1960*

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorised March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume LI, *number* 4 April 1958

Miniature number systems, BURTON W. JONES 226
Symbols and functions, IRVIN H. BRUNE 232
Facts, fancy, fabrications, and figures, M. H. AHRENDT 236
Can we continue to ignore these warnings? JOINT COMMITTEE ON ATOMIC ENERGY 240
The changing concept of trigonometry as a school subject, HERMAN ROSENBERG 246
Let's teach statistics, CLARENCE E. OLANDER 253
Finding two fractions having a special sum, P. H. NYGAARD 261
The teaching of mathematical infinity, W. M. PEREL 263
The Euler identity, ROBERT C. YATES 266
These eighth graders don't hate arithmetic, GEORGE R. ANDERSON 267
Time to stand together, WILLIAM G. CARR 270
On teaching dihedral angle and steradian, HOWARD FEHR 272
"Applications" in freshman courses: an example, ROBERT L. DAVIS 276

DEPARTMENTS

Historically speaking,—HOWARD EVES 280
Mathematics in the junior high school, LUCIEN B. KINNEY *and* DAN T. DAWSON 287
Points and viewpoints, DAYMOND J. AIKEN 292
Reviews and evaluations, RICHARD D. CRUMLEY 294
Testing time, ROBERT S. FOUCH *and* ROBERT KALIN 299
Tips for beginners, FRANCIS G. LANKFORD, JR., *and* JOSEPH N. PAYNE 303
Have you read? 231, 252, 262; *What's new?* 269

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Notes from the Washington office 308
Report of the Committee on Supplementary Publications 309
Registration reports on two recent meetings 311
Your professional dates 312

THE MATHEMATICS TEACHER is published monthly eight times per year, October through May. The individual subscription price of $3.00 ($1.50 to students) includes membership in the Council. Institutional subscription: $5.00 per year. Single copies: 50 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics*, 1201 Sixteenth Street, N.W., Washington 6, D. C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

Miniature number systems

BURTON W. JONES, *University of Colorado, Boulder, Colorado.*

Fields, rings, and groups! Some high schools will wish to experiment with the simpler ideas of modern algebra as possible mathematical "food" for bright youngsters.

THE NUMBER SYSTEMS with which we are most familiar contain infinitely many numbers. This article deals with systems which have only finitely many numbers, yet which have most of the important properties which we expect of a number system. These miniature number systems are not as bizarre as they might seem, for not only do they have connections with everyday experience, but they are in many respects simpler than the more familiar systems.

So that we may use the properties of our usual number systems as a guide in our new development, first consider the *natural numbers*, or *positive integers:* 1, 2, 3, · · · . This system has certain familiar properties:

1. Closure: $a+b$ and ab are natural numbers,
2. Commutative: $a+b=b+a$ and $ab=ba$,
3. Associative: $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$,
4. Distributive: $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$,

for all natural numbers a, b, c. In the system of positive integers, the equation $a+x=b$ is not solvable when $a \geqq b$. To avoid these exceptions, we introduce zero and the negative integers: -1, -2, -3, · · · , to get the enlarged system of *integers*. In this system the four properties above hold, and also $a+x=b$ is solvable for all integers a and b. In more advanced mathematics such a number system is called a *commutative ring*.

There is another important property of the integers which is often used but less often pointed out, namely: $ab=0$ implies that one or both a and b are zero. When this property holds for a commutative ring, it is called an *integral domain*. In the light of these properties we now consider a simple miniature number system.

A MINIATURE SYSTEM OF SEVEN NUMBERS

One of the simplest of number systems is that associated with the days of the week. Five days after Sunday is Friday, for instance. This can be made numerical by letting the numbers 0, 1, 2, 3, 4, 5, 6 correspond respectively to the days of the week, beginning with Sunday. Then "five days after Sunday is Friday" could be written $0+5=5$, which we would expect in any case. But "five days after Wednesday is Monday" has a somewhat more startling numerical representation: $3+5=1$. In general, the "sum" in this system is the remainder when the ordinary sum is divided by 7. We call this "addition (mod 7)" and see that the addition table is as follows:

+ (mod 7)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

We might check the three additive properties for this new system. The closure property holds, since only the seven numbers occur in the body of the table. The

commutative property holds, since the table is symmetric about the diagonal from the upper left to the lower right corner of the table. The associative property for addition (mod 7) holds, since $(a+b)+c=a+(b+c)$, and the remainders when each side is divided by 7 must be the same.

Also the equation $a+x=b$ is always solvable; this may be seen as follows. If a and c are given, we find $a+c$ from the table by looking at the entry in the table in the row numbered a and column numbered c. Hence, to solve $a+x=b$ we find the number b in the row numbered a and x is the column in which b occurs. For instance, to solve $5+x=2$, we look for 2 in the row numbered 5, see that it is in the column numbered 4 and hence that $x=4$. Such equations are always solvable because each of the numbers of the system occurs in each row. Another way of stating the same thing is that no number occurs twice in any row. (Why are the two statements equivalent?) Of course the same statements could be made about columns.

We may also multiply (mod 7). For instance, $2 \cdot 5=3$ (mod 7), since 3 is the remainder when 10 is divided by 7. By this means the multiplication table below may be constructed:

· (mod 7)	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Just as above, we may show that the closure, commutative, and associative properties hold for multiplication (mod 7) and that the distributive properties hold as well. Hence the system forms a commutative ring.

But we have two extra dividends. We can see from the table that if $ab=0$, then a or b or both must be zero. Hence the system is an integral domain. Moreover, $ax=b$ is always solvable when $a \neq 0$ because each row contains, after zero, all the numbers from 1 to 6 inclusive. For instance, to solve $5x=3$, we find 3 in the row numbered 5, see that it is in the column numbered 2, and hence that $x=2$ is the solution. Such a system is called a *field;* that is, if in a commutative ring, the equation $ax=b$ is always solvable when $a \neq 0$, the ring is called a field. In this case, it has a finite number of elements and hence is a *finite field.* Another name for this is *Galois Field.* It is often designated by GF(7). It behaves like the rational numbers (or real or complex numbers) in that properties 1. to 4. hold, and division except by zero is always possible.

However, our system fails to have a number of properties which the rational and real number systems possess. For example, there is no order relationship—we cannot say that any number is "less than" any other number, at least if "less than" has the properties we usually associate with this relationship. A formal proof of this statement would take too long here, but it can be made reasonable by observing that $2+x=3$ for $x=1$ and $3+y=2$ for $y=6$. Hence, either 2 would be both less than 3 and greater than 3, or else not both 1 and 6 would be positive. The former alternative would violate one of the fundamental properties of an inequality, and the latter would violate another, since $6=1+1+1+1+1+1$. The complex numbers form another example of a number system which forms a field but which is not ordered. On the other hand, the equation $x^2=3$ is solvable in the real or complex number system but not in the Galois Field GF(7).

A MINIATURE SYSTEM OF TWELVE NUMBERS

One might also consider a number system associated with the hours on a clock. As above, it may be seen that the addition and multiplication tables are:

+ (mod 12)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

· (mod 12)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11
2	0	2	4	6	8	10	0	2	4	6	8	10
3	0	3	6	9	0	3	6	9	0	3	6	9
4	0	4	8	0	4	8	0	4	8	0	4	8
5	0	5	10	3	8	1	6	11	4	9	2	7
6	0	6	0	6	0	6	0	6	0	6	0	6
7	0	7	2	9	4	11	6	1	8	3	10	5
8	0	8	4	0	8	4	0	8	4	0	8	4
9	0	9	6	0	9	6	0	9	6	0	9	6
10	0	10	8	6	4	2	0	10	8	6	4	2
11	0	11	10	9	8	7	6	5	4	3	2	1

It is easy to see that this system forms a commutative ring. But this system is not an integral domain since, for instance, $3 \cdot 4 = 0 \pmod{12}$, where neither 3 nor 4 is zero. Notice that $ax = b$ is not always solvable, either, since most of the rows of the above table contain repeated numbers. For instance, $3x = b$ is solvable only when b is one of 0, 3, 6, 9. The only nonzero values of a for which $ax = b$ is always solvable are 1, 5, 7, 11.

Finite fields

Consider the general number system (mod m), that is, the set of numbers $0, 1, 2, 3, \cdots, (m-1)$, where in addition and multiplication the remainder after division by m is taken as the sum or the product. It is not hard to see that addition and multiplication have the properties required of a commutative ring, no matter what positive integer m is. But if $m = rs$, where r and s are different from 1 and m, then $rs = 0 \pmod{m}$, with neither r nor s zero. Hence, *a number system* (mod m) *forms an integral domain if and only if* m *is a prime number.*

Furthermore, if m is a prime number p, $ax = b$ will always be solvable in the system for $a \neq 0$. This can be proved as follows. Designate the numbers in the row numbered a of the multiplication table by

$$0, r_1, r_2, r_3, \cdots, r_{p-1},$$

where $r_1 = a$, r_2 is the remainder when $2a$ is divided by p, r_3 is the remainder when $3a$ is divided by p, $\cdots$, r_{p-1} is the remainder when $(p-1)a$ is divided by p. To show that no two r's are the same, suppose, for example, that $r_3 = r_5$. Then $3a = kp + r_3$, $5a = tp + r_5$, and if $r_3 = r_5$, then

$$5a - 3a = tp - kp = (t-k)p.$$

Thus $2a = (t-k)p$, which implies that p divides 2 or a. It cannot divide 2, since 2 is positive and less than p; and it cannot divide a, since $a \neq 0$ in the system. Hence $r_3 \neq r_5$. In general, suppose $r_i = r_j$, with $i > j$. Then $ia = kp + r_i$ and $ja = tp + r_j$ im-

plies $(i-j)a=(k-l)p$. As above, p cannot divide a by hypothesis, and it cannot divide $i-j$, since $i-j$ is less than p and positive. This completes our proof. From one point of view, we have shown that if two entries in the row are the same, then a zero occurs in the row after the first entry, which is impossible.

We have thus shown the following important result: *the numbers* 0, 1, $\cdots$, $(m-1)$(mod m) *form a field if and only if* m *is a prime number* p. We designate these fields by GF(p).

Are these the only finite fields? To answer this question, let F be any finite field. It contains 0 and 1. Since it is closed under addition, it contains $1+1$, $1+1+1$, $1+1+1+1$, etc. Since F contains only a finite number of numbers, two such sums must be equal. Hence their difference would be zero and also another such sum. Hence there must be some smallest natural number n such that $1+1+1\cdots+1$ (n times) is zero. F then contains a number system (mod n) which would have necessarily the attributes of a field, and hence n would have to be a prime number p. (This number p is called the *characteristic* of the field.) If these are the only numbers in p, then $F=\mathrm{GF}(p)$, and we have the kind of finite field described above. If not, let x be some other number of F. Then, again from the property of closure, F contains the p^2 numbers $ax+b$ where a and b range over the numbers of GF(p). If these are all the numbers of F, then we have GF(p^2). If this is not the case, F contains some number y not expressible in the form $ax+b$, and hence the p^3 numbers $cy+ax+b$ lie in F, where c, a, and b range over the numbers of GF(p). So we may proceed to show that any finite field contains p^n elements for some prime p and some positive integer n. It is true that, except for a change in notation, the field is completely determined by p and n, but we shall not prove it here.

The above shows that there is some limitation on the number of elements in a finite field. It is true that for any prime number p and any natural number n, there is a finite field with p^n elements. We shall not prove this here, but will show a method for constructing such fields.

A construction of the field of complex numbers

By way of leading into the construction of finite fields, we first illustrate the method by showing a method of construction of the complex number field from that of the real numbers. Consider the set of binomials $ax+b$ where a and b are real numbers. We add these binomials in the usual way, but in multiplying we replace the product by its remainder after division by x^2+1. This is really a system of binomials mod (x^2+1). For example, $(x+\sqrt{3})(2x+5)=2x^2+(2\sqrt{3}+5)x+5\sqrt{3}$ $=2(x^2+1)+(2\sqrt{3}+5)x+5\sqrt{3}-2$. And hence we call $(x+\sqrt{3})(2x+5)=(2\sqrt{3}+5)x$ $+5\sqrt{3}-2$ mod (x^2+1). By this means, we make the set of binomials closed under multiplication mod (x^2+1), that is, the product of two binomials $ax+b$ with a and b real numbers is mod (x^2+1), another binomial of the same type.

It is easy to see that these polynomials mod (x^2+1) form a commutative ring as defined above, assuming that the real numbers have this property. To show that they form an integral domain, suppose $(ax+b)(cx+d)$ were 0 mod (x^2+1). Then $(ax+b)(cx+d)$ would have to be divisible by x^2+1. Since both the product and x^2+1 are quadratic,

$$(ax+b)(cx+d)=k(x^2+1)$$

for some real number k, which is impossible since x^2+1 has no linear factors with real coefficients.

Actually, these polynomials form a field. This will be established if we show that whenever a and b are not both zero, we can find real numbers c and d such that $(ax+b)(cx+d)=1$ mod (x^2+1). To show this, notice that

$$(ax+b)(cx+d)=ac(x^2+1)+(bc+ad)x+(bd-ac).$$

We then must show that

$$bc+ad=0$$
$$-ac+bd=1$$

is solvable for c and d, no matter what a and b are except $a=b=0$. The determinant of this pair of equations is a^2+b^2, which is not zero for any real a and b except $a=b=0$. Hence the pair of equations is always solvable, and we have proved our result: *the set of binomials* $ax+b$ mod (x^2+1) *form a field.*

This field is really the field of complex numbers, since $x^2=-1$ mod (x^2+1) and x takes the place of the usual i. One could form "another" field in a similar way using y^2+2 instead of x^2+1. Actually, this is the same as the complex number field, as may be seen by replacing y by $\sqrt{2}y$.

But one could not use the binomial x^2-1 in place of x^2+1. Why? Could a properly chosen cubic polynomial be used in place of x^2+1?

Construction of finite fields GF(p^n), $n>1$

For example, take $p=3$ and $n=2$. As above, consider the binomials $ax+b$ where a and b take on the values 0, 1, 2 (mod 3) and the product of two binomials is taken mod (x^2+1). For instance, $(2x+1)(2x+2)=4x^2+6x+2$. If we write the coefficients (mod 3), we get $x^2+2=(x^2+1)+1$. Hence mod $(3, x^2+1)$ the product is 1. Just as above, division is always possible since $a^2+b^2=0$ is possible (mod 3) only if a and b are 0 (mod 3). We thus have a field with 9 numbers. Notice that we could not have used x^2+2 in place of x^2+1 since in GF(3) $x^2+2=x^2-1$. Furthermore, if p were 5 instead of 3, we could not use x^2+1 since $x^2+1=(x-2)(x+2)$(mod 5), and our commutative ring would not be an integral domain.

To construct a field with 27 elements, we would have to find a cubic polynomial which has no factors in GF(3). Such a polynomial is x^3-x+1, for if it were factorable, it would have a linear factor $x-a$ and hence would be zero when x is replaced by a (mod 3) for a one of 0, 1, 2. But for all these values, x^3-x+1 has the value 1, which is not zero.

The smallest field consists of the numbers 0 and 1 (mod 2). We could not use the binomial x^2+1 to construct a field with four elements, since $x^2+1=(x+1)(x+1)$ (mod 2). But we could use the polynomial x^2-x+1, which is zero for neither $x=1$ nor $x=0$.

These examples show that the constructibility of a finite field GF(p^n) depends only on the existence of a polynomial of degree n with coefficients in GF(p) which has no factors of lower degree, that is, an *irreducible polynomial.* This is not very difficult to prove, but we attempt no proof here.

Another important property not proved here is: every finite field contains a number which "generates" the field, that is, whose powers are all the nonzero numbers of the field. This is illustrated in the tables below.

Reference

Garrett Birkhoff and Saunders MacLane. *A Survey of Modern Algebra,* Chapters 1 and 15. New York, Macmillan Company, 1941.

TABLE 1

GF(8): trinomials ax^2+bx+c mod (x^3+x+1) with coefficients in GF(2)

n	0	1	2	3	4	5	6	7
x^n	1	x	x^2	$x+1$	x^2+x	x^2+x+1	x^2+1	1

TABLE 2

GF(9): binomials $ax+b$, mod (x^2+1) with coefficients in GF(3)

n	0	1	2	3	4	5	6	7	8
$(x+1)^n$	1	$x+1$	$2x$	$2x+1$	2	$2x+2$	x	$x+2$	1

TABLE 3

GF(27): TRINOMIALS ax^2+bx+c, mod (x^3-x+1) WITH COEFFICIENTS IN GF(3), TAKEN AS 0, 1 AND -1

n	0	1	2	3	4	5	6	7	8
x^n	1	x	x^2	$x-1$	x^2-x	$-x^2+x-1$	x^2+x+1	x^2-x-1	$-x^2-1$
n	9	10	11	12	13	14	15	16	17
x^n	$x+1$	x^2+x	x^2+x-1	x^2-1	-1	$-x$	$-x^2$	$-x+1$	$-x^2+x$
n	18	19	20	21	22	23	24	25	26
x^n	x^2-x+1	$-x^2-x-1$	$-x^2+x+1$	x^2+1	$-x-1$	$-x^2-x$	$-x^2-x+1$	$-x^2+1$	1

Have you read?

BIRD, OTTO. "Science and Mathematics in the Liberal Arts Curriculum," *Journal of General Education*, January 1957, pp. 24–29.

Here is presented a point of view new to me. The author defines "Liberal Arts" as "understanding and skill with signs and symbols so as to achieve knowledge." He concludes that most liberal arts colleges are really not this, but only "liberal knowledge" colleges.

In line with his definition, the writer shows how science and mathematics are vital to a liberal arts curriculum. For example, he points out that mathematics is a language, that demonstration and axiomatization form one of the highest achievements of language, that mathematics provides for thorough analysis, eliminates ambiguity, increases precision, provides for proof, and establishes relationships regardless of environment.

Mathematics is the language of reasoning, it contributes to all the general ideas of the intellectual. Every person should study mathematics, physics, and biology if he hopes to have what a "Liberal Arts Curriculum" is supposed to provide. This is an article you must read.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

BROWN, JOHN A. (compiled by). "Promising Practices in Mathematics Teacher Education," *School Science and Mathematics*, January 1958, pp. 25–40.

Here is a summary of a series of reports given at the recent Regional Conference on Science and Mathematics Education. Materials covered are "Trends in Teacher Education." You will be interested to note the trend toward more abstraction to aid thinking, the increase in inferential statistics, some increase in emphasis on logic, and in general a more thoughtful approach to secondary mathematics. All this affects the program of teacher education. Modern algebra should play a more important role, statistics must be included in training programs, and the nature of the structure of mathematics must be taught. This also means in-service programs must be provided for those of us who are beyond the normal training years. Another change that is needed is in the preparation of those teaching junior-high-school mathematics. There are many stimulating ideas presented in this article, and I am sure you will want to read it.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

COHEN, JOHN. "Subjective Probability," *Scientific American*, November 1957, pp. 128–138.

Here is a stimulating article you will want to read. It brings probability into the everyday life of each of us. Such examples are given as probability of success and failure; making choices with high and low probabilities; the language of probability in words like "most," "many," and "some"—and of most importance, the probability of many answers to questions which occur in life.

You will be interested in the information about probability elementary pupils possess, for example, they seem to pass through four stages. Up to ten years of age, children merely guess; in the second stage, children have some vague comparisons; the third stage brings a more precise analysis of relationship; and in the fourth stage, children objectify.

It is interesting to learn that personality seems to have an effect and that experienced people require higher probability of success. These and many other items are mentioned. I am sure your curiosity will be aroused.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Symbols and functions

IRVIN H. BRUNE, *Iowa State Teachers College, Cedar Falls, Iowa.*

Some common words, often worn fuzzy in ordinary usage, need more than ordinary care in mathematics.

Tinkling symbols

Teachers of mathematics are symbol-minded people. Many of their pupils, however, acquire this desirable behavior painfully, if at all. Not that they cannot mouth the symbols! But for them to appreciate the meanings of the symbols is another thing.

> They like the words 'infinity,'
> 'Divergent,' and 'scalene.'
> They toss them off with nonchalance,
> But don't know what they mean!

The present article looks at a few words. They are simple words, appearing often in ordinary speech. But that is the very reason why they can cause trouble in the classroom. One of them merits 163 definitions in an unabridged dictionary, and this does not include eighty or so idiomatic uses. The words are those such as 'set', 'variable', 'constant', 'pair', 'relation', and 'function'.

A significant question

Suppose we look at some strange symbols: 'Was ist Mathematik?' Apparently the marks are words. One can quickly learn to repeat them. Through drill, one can easily learn to write them. One can puzzle out what they mean. In English, the words become 'What is mathematics?'

Weightier, though, than their deciphering is the question they symbolize. Just what is mathematics? Doubtless any serious study of mathematics augments appreciably the answer each student continually develops for himself.

We know, indeed, that answers abound to the question, What is mathematics? Shades and distinctions among these answers provide an infinitude of variety. And, since mathematics itself continues to grow, the answers multiply.

From the four or so scholarly pages of Courant and Robbins[1] to the terse, possibly overly-simplified, statements of numerous other students of the subject, a tremendous range of ideas exists. Stated with single symbols, these ideas inevitably include *number*, *measure*, *relation*, and *proof*.

Successful teachers of mathematics increasingly stress such big ideas. Our objective in the paragraphs that follow is to focus mostly on one item among these four, namely, *relation*. Indeed, we plan to concentrate on a special case of relation, the kind we call *functions*.

Symbols

Obviously we shall employ symbols. For in mathematics, symbols sweep the subject. They pervade its assumptions; they facilitate its operations; they suggest its applications. The very words we employ are symbols. The sentences we write are symbols. The marks, scrutable or inscrutable, we scratch are symbols. In mathematics or outside mathematics, symbols enable people to think.

[1] Richard Courant and Herbert Robbins, *What Is Mathematics?* (New York: Oxford University Press, 1941), pp. xv–xix.

Symbols and sets

The search for first principles interests teachers as well as philosophers. What are some building blocks in mathematics? One basic notion bears the name 'set'. Children in school handle sets of objects and learn to deal with expanding sets of numbers. Algebra stems from a set of postulates. Geometry treats of sets of polygons. A function is a set of ordered number pairs.

'Set', in ordinary usage, refers to a collection of objects that go together, or complement one another. A set of golf clubs and a set of teeth are examples. In mathematics a set is any collection of objects.

Besides listing some or all of the elements of a set, we can describe a set by words or delimit it by a rule. One can talk about the set of rectangles that enclose no more than 100 square feet. Or people lick stamps according to the rule: Three cents for each ounce or fraction thereof. Such statements suggest sets—sets of number pairs, the first number in each pair being the number of ounces, and the second number being the price. Thus words specify sets.

So too, do tables. If 6 cats eat 6 rats in 6 minutes, some elements in a set of number triples might be:

Cats	Rats	Minutes
6	6	6
6	1	1
6	2	2
6	3	3
6	4	4
6	5	5
.	.	.
.	.	.
.	.	.
6	10	10
.	.	.
.	.	.
.	.	.
x	100	100

Accordingly, the answer to that patently practical question, "How many cats, under the present assumption, would be needed to eat 100 rats in 100 minutes?" would be 6.

Symbols and constants

Indeed, the table's first column, which is itself a set of symbols representing numbers of cats, merits another look. It contains but one element. If we let '*c*' represent the number of cats in any line in the table, then '*c*' may assume only one value. The symbol '*c*', in other words, is a *constant*.

Here a firm N.B. is in order. We should note well that '*c*' symbolizes any element in the set of numerals representing cats. It is not an abbreviation for *cats*. Neither is it shorthand for *constant*. Confusion sometimes reigns in pupils' minds when they memorize a rule such as '$A = lw$'. Besides failing to understand the mysterious shift, in this case from linear units to square units, pupils may miss the major point that a variable, such as '*w*', fills a place for a member of a set of numerals. Such a symbol lets the rule apply for all possible widths. Since abbreviations are not essential, we might as well rejoice that good, old-fashioned '*x*' endures. Although it abbreviates few, if any, words, it serves well to symbolize any element in a specified set.

Symbols and variables

We should note, too, that, whereas the set of numbers of cats possesses but one member, the sets labeled *Rats* and *Minutes* contain several elements. The latter sets, in fact, could contain many members, provided, of course, that the cats in our situation could stand the gorging.

From this, if '*r*' represents the numbers in the set headed *Rats*, and if '*m*' represents the numbers in the set headed *Minutes*, then '*r*' and '*m*' each may take on any one of several values; they represent various values. They are, in this sense, *variables*.

Students of symbols sicken at statements to the effect that "variables are

numbers that vary." We may let a symbol represent any one of various numbers, but each of those numbers itself is fixed.

Mathematical symbols versus ordinary symbols

Here, as in numerous other instances in mathematics, words in mathematics differ sharply from the same words in ordinary usage. Near a Midwestern motel appears a sign emphasizing the words: 'Variable Rates'. Now, doubtless, this sign hints at mathematics. But, doubtless too, 'variable' here is not the technical 'variable' in mathematics. The connotations emanating from the word 'variable' on the sign probably include:

1. Rates that shift with the seasons.
2. Rates set according to the tourist's make of automobile.
3. Rates appropriate (in the operator's eye) to each of several kinds of accommodations.
4. Rates that lessen when the tourist season slacks off, when the hour grows late, or when the attendant becomes weary.

Yet, however much 'variable' in ordinary conversation may imply change, 'variable' in mathematics denotes a symbol (whether it is a suggestive initial is immaterial) that represents any element in a set of elements.

Symbols and relations

Sets underlie much of modern mathematics. So does another concept, the idea of pairing. Young children pair and compare. They match books with people, cookies with children, or places at table with persons to be served.

Pairings of seats and people to be seated provide many possibilities. If there are 5 persons and 5 chairs, each chair could accommodate any of 5 different people. Hence, there would be 25 different pairs possible. Such a set is the Cartesian product of the two sets, the set of people and the set of chairs.

People's Names	Chair Numbers 1	2	3	4	5
Tom	(1,T)	(2,T)	(3,T)	(4,T)	(5,T)
Jane	(1,J)	(2,J)	(3,J)	(4,J)	(5,J)
Dick	(1,D)	(2,D)	(3,D)	(4,D)	(5,D)
Sue	(1,S)	(2,S)	(3,S)	(4,S)	(5,S)
Harry	(1,H)	(2,H)	(3,H)	(4,H)	(5,H)

Furthermore, the places could be named (number names would serve), and their names could be paired with people's names. Although confusions among such pairs wouldn't be likely (there was a time, though, when children in Siam received numbers for names), the pairs could be *ordered*, with each first element being a number and each second element being a person's name.

This set, or any set, of ordered pairs is a relation. This, of course, is a broad usage. It is more precise, though, than the metaphysician's statement: "Any sort of connection which is perceived or imagined between two or more things, or any comparison which is made by the mind is a relation."

Perhaps a use for the various seating pairs possible in the example we cited does not readily come to mind. But, depending on the occasion, certain subsets may be useful.

So relations often are subsets. The sets $\{1, 2, 5\}$ and $\{-1, 11, 12\}$ provide nine pairs: $\{(1, -1), (1, 11), (1, 12), (2, -1), (2, 11), (2, 12), (5, -1), (5, 11), (5, 12)\}$. Although an application for this set of ordered pairs may or may not arise, the subset $\{(1, 12), (2, 11), (5, -1), (1, 11), (2, 12), (5, 12)\}$ might represent early-morning temperatures in two localities. And the subset $\{(1, 12), (2, 11), (5, -1)\}$ is one of several possible subsets worthy of special consideration because in them no specific first element in a pair appears elsewhere paired with a different second element. (See the following section.)

In ordinary speech, the word 'relation(s)' represents six or eight ideas. It may suggest an account, one's kin, or foreign affairs. Or, it may indicate a property common to two or more things.

As a symbol for a set of ordered pairs, however, 'relation' is clear-cut.

Symbols and functions

Among the pairings previously considered between numbers and persons' names, the name *one* as a first element could be followed by Tom or Jane or Dick or Sue or Harry—any one of five names. Sometimes, though, a relation in which no two pairs having the same first element and different second elements is useful. Thus, for a time at least, {(one, Tom), (two, Jane), (three, Dick), (four, Sue), (five, Harry)} may be an efficient table plan in a mathematics classroom. Chair One is paired with Tom only, and so on. Such a set of ordered pairs illustrates a *function*. The subset $\{(1, 12), (2, 11), (5, -1)\}$ referred to previously also indicates a function.

According to this usage, one's weight is not strictly a function of one's height. For, given a definite height in inches, exactly one weight in pounds does not necessarily follow. At, say, 72 inches, many different weights are possible.

As with the word 'relation', the symbol 'function' elicits many ideas in ordinary speech. About a dozen meanings, including activity, festivity, and propensity, can confuse pupils in mathematics, unless the idea is clear that, given a first element in a pair, there is no ambiguity about the second element.

Résumé

Mathematical *denotations* as opposed to less precise *connotations* have been our objective. We have considered a few symbols. The word 'mathematics' is itself a significant symbol. We considered *variables*, each of which may represent unspecified members of a set. If the set has exactly one member, the symbol representing that member is a *constant*. Although we did not treat them in detail, *unknowns* and *parameters* are variables. They, too, refer to unspecified members of a set of applicable numbers. If a condition involving an unknown boils down to '$x-x=2$', the set of values x represents is evidently empty; the set is the *null set*.

From a quadruple of major concepts, we selected the idea of relation. A *relation* is a set of ordered pairs, often a subset of all pairs possible when the elements of two sets are matched. A further special case is a *function*, in which each first element corresponds to exactly one second element. To designate and describe relations, we have largely employed verbal symbols. But graphs, tables, and formulas also represent relations and functions.

Insight is the principal thing; therefore we seek insight. Teachers of mathematics realize that clear-cut symbols enhance their own insights and their pupils' insights. Tables display them, graphs portray them, formulas condense them, series spell them out, and differential equations pin them down. These modes are concise and precise. Words, alone, can delude. Teachers need continually to distinguish between concepts and words, and to differentiate between ordinary usage and technical usage. A pupil's glibness cannot substitute for his insight.

"The *principles of substitution* is one of the most important elements in the science; it is this that gives to algebraic calculation its immense power and almost endless capability of development."—*Taken from G. Chrystal's historically famous text*, Algebra: An Elementary Textbook.

Facts, fancy, fabrications, and figures

M. H. AHRENDT, *Executive Secretary, NCTM, Washington, D.C.*

Here are authoritative figures on enrollments in high school mathematics classes. Note the amount of mathematics carried by scholarship-winning students.

FOR A NUMBER OF YEARS the debate over the status of mathematics and science education has been going on. The schools have been criticized, vilified, defended, and lauded. The trouble is that so much of what has been said has been based on inaccurate data or on misinterpretations of accurate data.

It seems especially appropriate at this time that the facts be made available. Below is a collection of reliable statistical information. Although this information is and has been available to all interested persons, it has not been found, or if found has not been used, by the critics of the schools.

ENROLLMENTS, OPPORTUNITY, AND INTEREST ARE HIGHER THAN HAS BEEN CLAIMED

Table 1 shows, for example, how erroneous was the recent statement by a nationally-known individual that only 12 per cent of high school students study elementary algebra. The facts are that about two-thirds of the ninth-grade students now study algebra, about 42 per cent of the sophomores study plane geometry, and nearly one-third of the juniors take intermediate algebra. The enrollments in science courses are also much higher than the critics have admitted.

Because of small enrollments and inadequate facilities, some students are deprived of the opportunity to take advanced courses in mathematics and science. This is a deficiency over which the public should justifiably be aroused, and funds should be supplied to make an adequate education available to all. But because the schools involved are in general small, the number of students affected is not as great as has been claimed. Correct information about these shortages is given in Table 2.

TABLE 1

ENROLLMENTS IN CERTAIN COURSES EXPRESSED AS THE PERCENTAGE OF PUPILS IN THE GRADE WHERE THE COURSE IS USUALLY OFFERED, 1956–57 (*Source: U. S. Office of Education*)

Course	Grade Offered	Per Cent	Course	Grade Offered	Per Cent
General Mathematics	9	43.1	General Science	9	67.0
Elementary Algebra	9	67.0	Biology	10	75.5
Plane Geometry	10	41.6	Chemistry	11	34.6
Intermediate Algebra	11	32.2	Physics	12	24.3
Trigonometry	12	9.2			
Solid Geometry	12	7.6			

TABLE 2

PERCENTAGE OF PUPILS DENIED OPPORTUNITIES TO TAKE CERTAIN COURSES IN THE FALL OF 1954. BASED ON TOTAL ENROLLMENT IN GRADE AT WHICH COURSE IS USUALLY TAUGHT (*Source: U. S. Office of Education*)

COURSE	GRADE OFFERED	PER CENT	COURSE	GRADE OFFERED	PER CENT
Plane Geometry	10	6.8	Biology	10	3.3
Advanced Mathematics	12	10.9	Chemistry, Physics	12	5.8

TABLE 3

AMOUNTS OF CREDIT IN HIGH SCHOOL SCIENCE AND MATHEMATICS EARNED BY SCHOLARSHIP-WINNING STUDENTS (*Source: Study by M. H. Ahrendt, reported in* NEA Journal)

	AVERAGE NUMBER OF UNITS OF CREDIT EARNED			PER CENT OF TOTAL GRADUATION REQUIREMENT DEVOTED TO MATHEMATICS AND SCIENCE	PER CENT OF STUDENTS INDICATING A VOCATIONAL PREFERENCE FOR A SCIENTIFIC FIELD
	MATHEMATICS	SCIENCE	COMBINED		
1953 winners	3.56	2.95	6.51	40.6	55
1956 winners	3.64	2.94	6.58	41.1	60
Total	3.60	2.95	6.55	41.0	58

The assertions have been made that our most able students are no longer interested in science and mathematics, that they take the easy courses, and that engineering and the sciences are being deprived of our best brains. The writer had the opportunity to study in detail the amount of credit earned and the vocational preferences indicated by scholarship-winning students in the Annual National Honor Society Program. The figures in Table 3 indicate that our best students take full programs in mathematics and science and lean strongly toward scientific occupations.

THE SHORTAGE OF ADEQUATELY TRAINED TEACHERS IS MORE CRITICAL THAN IS GENERALLY REALIZED

The preceding three tables tell an interesting story about the demand and supply for teachers of mathematics. Table 4 shows that there was a rapidly dwindling supply of college graduates adequately prepared to teach mathematics from 1950 through 1955, with some improvement in the supply situation in 1956 and 1957. Note that 2544 persons graduated from college in 1956 with adequate preparation (majors in mathematics and sufficient education courses) to enter the profession as teachers

TABLE 4

COLLEGE GRADUATES WHO ARE PREPARED TO TEACH MATHEMATICS, 1950–57 (*Source: NEA Research Division*)

YEAR	COLLEGE GRADUATES WITH MAJOR PREPARATION IN MATH	PER CENT CHANGE FROM 1950
1950	4618	—
1951	4118	10.8
1952	3142	32.0
1953	2573	44.3
1954	2223	51.9
1955	2155	53.3
1956	2544	44.9
1957	2892	37.4

of mathematics. Table 5 shows that about two-thirds of these, or 1700 persons, actually entered teaching. Thus 1700 adequately prepared teachers were available to fill the vacancies during the 1956–57 school year.

TABLE 5

PERCENTAGES OF THOSE PREPARED TO TEACH SCIENCE AND MATHEMATICS WHO ENTERED TEACHING IN 1956 (*Source: NEA Research Division*)

FIELD OF PREPARATION	NUMBER PREPARED TO TEACH	PER CENT WHO ENTERED TEACHING
General Science	1889	64.2
Biology	1706	58.5
Chemistry	521	54.3
Physics	204	47.3
Total Science	4320	59.7
Mathematics	2544	66.3

How did this supply of 1700 compare with the demand for mathematics teachers? Table 6 shows that it was necessary to employ during 1956–57 nearly 5000 persons who taught mathematics more than half time and therefore needed to have their major preparation in mathematics. Thus the colleges were able to supply only about one-third of the new teachers required. It seems obvious that an overwhelming majority of the new teachers of mathematics being employed are inadequately prepared.

TABLE 6

NEW HIGH SCHOOL TEACHERS OF MATHEMATICS (*Source: NEA Research Division*)

EMPLOYED IN YEAR	FULL-TIME MATH	MORE THAN HALF-TIME MATH	LESS THAN HALF-TIME MATH	TOTAL
1	2	3	4	5
1952–53	1638	1485	1380	4503
1953–54	2112	1676	1512	5300
1954–55	2300	1618	1292	5210
1955–56	2723	1750	1515	5988
1956–57	3044	1905	1712	6661

Tables 5, 7, and 8 tell a similar story about teachers of science. In 1956 the colleges graduated 4320 persons adequately prepared to teach science. About 60% of these, or 2600, actually entered teaching. But the number of new teachers employed during 1955–56 to teach more than half-time science, and who therefore needed to have their major preparation in science, was 5534. Thus the colleges were able to supply less than one-half the actual demand.

TABLE 7

COLLEGE GRADUATES WHO ARE PREPARED TO TEACH SCIENCE, 1950–57 (*Source: NEA Research Division*)

YEAR	COLLEGE GRADUATES WITH MAJOR PREPARATION IN SCIENCE	PER CENT CHANGE FROM 1950
1950	9096	—
1951	7507	17.5
1952	5246	40.3
1953	4381	51.8
1954	3641	60.0
1955	3754	58.7
1956	4320	52.5
1957	5044	44.5

TABLE 8

NEW HIGH SCHOOL TEACHERS OF SCIENCE (*Source: NEA Research Division*)

EMPLOYED IN YEAR	FULL-TIME SCIENCE	MORE THAN HALF-TIME SCIENCE	LESS THAN HALF-TIME SCIENCE	TOTAL
1	2	3	4	5
1952–53	3009	1456	2503	6968
1953–54	3264	1544	2554	7362
1954–55	3514	1372	2464	7350
1955–56	4152	1725	2587	8464
1956–57	3653	1881	2810	8344

NOT ENOUGH PERSONS ARE CONTINUING IN HIGHER EDUCATION

Table 9 shows the rapid growth in number of graduates from secondary

TABLE 9

GRADUATES, HIGH SCHOOLS AND COLLEGES, PUBLIC AND PRIVATE, 1948–1959 (1957–1959 ESTIMATED)
(*Source: U. S. Office of Education*)

YEAR ENDED	HIGH SCHOOL GRADUATES	COLLEGE AND UNIVERSITY GRADUATES: Bachelor's or first professional degree	Master's or second professional degree	Doctor's degree
1	2	3	4	5
1948	1,189,909	271,019	42,400	4,188
1950	1,199,700	432,058	58,183	6,633
1952	1,196,500	329,986	63,534	7,683
1954	1,276,100	290,825	56,788	8,995
1955	1,281,400	285,138	58,165	8,837
1956	1,318,700	325,000	57,400	8,270
1957	1,358,600	347,000	65,900	8,460
1958	1,392,200	392,000	70,900	8,380
1959	1,454,400	421,000	80,700	9,750

1 IN 4 · 1 IN 5 · 1 IN 7

schools and colleges. This table makes it possible to discover the pattern involved in the progression of students toward higher degrees. The number of students, 1,196,500, graduated from high school in 1952 makes the best base with which to compare the number of bachelor's degrees earned in 1956, since the 1952 high school graduates form the major source from which the 1956 college graduates came. Similar relationships, as indicated in the table, connect the numbers of bachelor's, master's, and doctor's degrees earned. It seems clear that according to the present pattern, about one high school graduate in four will graduate from college, one college graduate in five will earn a master's degree, and only one master's degree holder in seven will complete his doctorate. Of prime concern to our nation is the need to increase the numbers of our able students who continue their education through college and graduate school.

The above discussion points out only some of the interesting facts that can be gleaned from these tables. Persons sincerely interested in education readily admit that there are shortcomings, inadequacies, and failures in our present educational system. They are probably more interested than are even the critics in finding solutions to our problems. They also are convinced that no real progress can be made unless the facts are known and used.

In 1886, G. Chrystal in his famous text *Algebra: An Elementary Textbook* defined a function as "*A rational integral algebraical function is the algebraical sum of a series of integral algebraical terms.*" Thus, $3x^3+3xy+2y^3$ is a function of x and y.

Can we continue to ignore these warnings?[1]

JOINT COMMITTEE ON ATOMIC ENERGY.

On July 12, 1957, the Congressional Committee on Atomic Energy reported: A Shortage of Scientific and Engineering Manpower.

Introduction

IN THE PAST DECADE the United States has taken enormous strides in technological development. Whole new vistas have been opened as a result of these advances and consequently the demand for skilled talent in almost all fields of endeavor is unprecedented. This has been particularly evident in the fields of science and engineering where shortages in some instances have reached the critical stage. At a time when science and technology are the key both to military superiority and to economic strength, the popular consciousness of the immediate and urgent need for men and women to man the laboratories, the drawing boards, and the production lines is essential and understandable.

It is important to recognize at the outset that these serious shortages are not a sign of failure but an indication of startling success. A way of life has been created in which the demands made for superior skills in every field have become so tremendous that growth and expansion have become constant elements of the normal pattern. A society has been created in which progress is not simply a matter of occasional spurts and short-term crises. Rapid progress has become a normal part of our national life, and the skilled manpower to keep up with this steadily and rapidly advancing march of progress will be in demand for years to come.

But these considerations should not becloud the fact that the shortage of skilled talent in this country is a very real problem and one with which we must come to grips if we are to maintain the present rate of growth which is so essential to the Nation's strength and well-being. We are, in essence, engaged in a battle for brainpower. The price of continuing progress and prosperity is an unceasing effort to develop our intellectual resources to the maximum possible extent.

A sense of added urgency has been given to the manpower problem in this country by disturbing reports that the Russians have launched a massive educational program to train scientists and engineers for use in Russia and, more ominously, for export to the uncommitted areas of the world. Few people will deny that the immediate need—vis-à-vis Russia—is indeed serious. But this is only one aspect of the total picture. The shortage of trained and educated manpower would be equally pressing without the Russian threat, although it would perhaps not be as readily recognized. To that extent the pressure created by the Soviet danger may even be beneficial.

Among the areas most vitally affected by the manpower shortage has been the field of nuclear energy which is emerging from an era of exclusively military application to one where peaceful uses are

[1] This report appeared as an advertisement in the *New York Times* of November 26, 1957, paid for by Judy Bond, Inc., and reprinted with their permission.

assuming an ever increasing importance. With this shift in emphasis toward peaceful applications has come a growing demand for trained technicians, a demand which can only partially be met by available sources of talent. The resulting shortage has been characterized by Commissioner Willard F. Libby of the Atomic Energy Commission as the greatest single deterrent to progress in our atomic energy program.

Impressed with the importance of developing a sound, coherent policy aimed at alleviating these shortages and providing for future needs, the Research and Development Subcommittee scheduled a series of public hearings in April, designed to determine the nature of the problem and to develop recommendations for corrective action. Five days of hearings were held during which nearly 500 pages of testimony was received from 32 witnesses, most of whom are leading experts in the fields of education, science, industry, labor, and Government.

A number of important points were developed during the hearings but two major themes kept recurring: (1) that the shortage of scientists and engineers is only part of a larger shortage of skilled manpower throughout our economy and that our interests will best be served in the long run by expanding our overall supply of trained individuals; and (2) that there has been a rapid deterioration in the quality and quantity of mathematics instruction in our high schools in recent years and that vigorous action is imperative to restore this important subject to its rightful place as a fundamental part of the curriculum, and as an indispensable element in preparation for college level studies in science and engineering. As a means of improving the quality of mathematics teaching and of arousing interest among students and their parents in improved mathematics courses, the committee believes that special consideration should be given to the recommendation by Prof. I. I. Rabi of Columbia University for the establishment of a Federal mathematics scholarship award program. (See major recommendations.)

The nature of the problem

It is almost universally conceded that there is a critical shortage of technically trained manpower in this country. But the nature of the crisis is not generally understood and the remedies frequently suggested tend to deal with the superficial and immediate problems rather than the more basic causes of the shortage. In this respect they threaten, in the long run, to create more serious crises.

The fundamental problem is not that the United States lacks manpower in one limited, clearly separable area. Yet, because the manpower shortage is too often treated for its symptoms rather than for its organic causes, the current fashion is to tackle it either piecemeal or by short-range, shock treatment.

This is understandable. The country is —and needs to be—in a position of alert. The United States considers itself as opposed ideologically, and possibly militarily, by a powerful combination of Soviet-bloc nations. The United States, moreover, is aware that it is the core and rallying point of the free nations and the people of this country are conscious of the pressing need for more technical manpower to maintain our military and economic strength.

Modern defense is total. It cannot be achieved in the medieval manner of putting up a few strong, scattered forts. It demands strength throughout the entire fabric of national life. It requires the kind of *cohesion and total national well-being that no military or peacefully competitive contest of the past has ever demanded.*

The important new fact of national life is that the shortages of educated, skilled brainpower extend to all areas—from the drawing boards to distribution; from engineering to management; from the laboratory to the classroom. There is an unprecedented demand for more skilled, educated

brains. The demand will increase for many years to come. Therefore, the shortage will become more critical.

But it would be tragic if—as the committee was warned—the United States is stampeded into short-sighted action. The committee was reminded of widespread forecasts made shortly after World War II which predicated an oversupply of engineers and, at least by implication, encouraged high school guidance counselors to steer young people into other vocational channels. These estimates were not inaccurate—at the particular time and based on the immediate, short-range picture. But they were tragically wrong in terms of the future.

The point, therefore, is that the critical shortages of manpower should be considered a total shortage of trained individuals in every field, and that in our efforts to remedy short-term deficiencies we should be careful not to create imbalances which will threaten our overall strength in the long run.

Major recommendations

During the course of its hearings the committee was impressed with the many excellent recommendations for remedial action. These deserve the most careful study and the committee hopes that a thorough analysis of them will be made in the next Congress with a view to developing a concrete program of action.

The principal requirements are (*a*) a stronger program for identifying the ablest students in our high schools at an early age and presenting them with challenging courses to stimulate their minds; (*b*) greater encouragement of students with high ability to continue their education beyond high school and removal of economic barriers; (*c*) taking all necessary steps to improve the supply of high-quality teachers at the high school and college level; (*d*) better utilization of talent presently available in industry, the Armed Forces, and education; (*e*) broadening in-service training and conversion of available talent in industry, in co-operation with educational institutions and others.

Listed below are a few of the major recommendations made to the committee that illustrate the types of approach which might assist in meeting these requirements:

(1) *Establishment of a Federal mathematics scholarship award program* to provide a substantial cash award to any high school graduate who passes an examination, at college entrance level, in mathematics. The examination would be open to all and there would be no compulsion to pursue studies in science. Chief objective of the award would be to arouse greater interest in high school mathematics as an important part of general education and an indispensable base for college-level study in science and engineering. For those who go on to college another cash award would be provided at the end of freshman year upon successful completion of a course in calculus.

(2) *Earlier identification of potentially ablest students.*—At present only 4 out of 5 students in the top quarter of their class finish high school. Only 2 out of 5 in the top quarter go on to college. Many with high potential simply do not enroll in college preparatory programs. In order to recoup this loss of potential talent it is proposed that statewide testing start at the 8th or 9th grade level rather than the 12th. This would help insure that the ablest students don't drop out of school and that they take the kind of courses necessary for college entrance. Increased emphasis should also be placed on intellectual rigor for the ablest students and challenging programs provided.

(3) *Increased pay for high school teachers and greater flexibility in salary scales.*—There was virtually unanimous agreement that substantial increases must be made in the pay of high school teachers if we are to attract and maintain the type

of qualified people we need. This is particularly true in the case of science and mathematics teachers who are being offered up to double their school salaries by private industry. In terms of actual purchasing power, the average salary of high school teachers is no greater today than it was in 1904. Also needed is abolishment of rigid salary scales which do not differentiate between the most able teachers and the less able. The present system acts as a source of discouragement to talented young teachers entering the profession and encourages mediocrity. In addition, encouragement should be given to expansion of summer institutes, such as those being conducted by the National Science Foundation, to provide refresher courses for high school mathematics and science teachers.

(4) *Federal contribution to private scholarship foundations.*—Federal funds could be used to supplement scholarships offered by private organizations such as National Merit Scholarship Foundation which provides 500 four-year college scholarships annually on the basis of a general nationwide competitive examination. Main advantage of the plan is that foundations are already going concerns and minimum time would be lost in putting Federal funds to work. It is interesting to note in this connection that nearly two-thirds of the boys selected for the national merit scholarships this spring indicated a career choice of engineering or scientific research. Consideration should also be given to supplementing teaching fellowships.

(5) *Establishment of a national educational reserve.*—To help alleviate the shortage of qualified mathematics and science teachers in our high schools it is proposed that a national educational reserve be set up comprising qualified teachers of mathematics, physics, chemistry, engineering, and related subjects, to be drawn from the ranks of private industry. Under this plan industry would release with full pay for at least a year a reasonable number of men and women for teaching assignments in nearby high schools, particularly in the upper grades. The reserve could also mobilize qualified persons of retirement age for part-time teaching and draw on qualified people willing to teach in night schools without taking leave from their industry jobs.

(6) *Better utilization in private industry.*—Industry should be encouraged to abolish the system of hiring excess numbers of our top young science and engineering graduates when there are actually no job openings. This practice is employed to ensure future availability in case of need but is wasteful of scarce talent and self-defeating in the long run because of the large turnovers which result. Greater efforts should also be made within industry to see that the full talents of scientists and engineers are being exploited and that routine work is left to less skilled personnel. In addition, on-the-job training and night courses for employees in science and engineering should receive greater emphasis.

(7) *Better utilization in the armed services.*—Despite improvement in utilization of scientists under the Armed Forces' scientific and professional personnel program (SPP), there are still some 5,000 or more specialists in the service who are "lost" in the enlisted ranks doing routine chores such as KP duty. Vigorous efforts should be made to locate these highly skilled men and action taken toward either assigning them duties and responsibilities commensurate with their scientific talents, or providing for their early release from the compulsory 2-year tour of duty. Considerable improvement can also be made in the ROTC program. At present, little attempt is made to assign young officers with scientific and engineering backgrounds to duties for which they have been prepared. The compulsory

2-year tour of duty for officers with these skills is largely wasted time and often results in failure of the exceptional student in the ROTC to continue his studies for an advanced degree. Perhaps the setting up of a separate scientific corps within the armed services would help insure more intelligent utilization, both in enlisted and officer ranks. Or consideration might be given to creation of a scientific personnel board with review power over both Selective Service and over use of scientists in the service.

(8) *Aid to universities in training technical assistance representatives.*—In view of the vital worldwide competition of the Communist philosophy for the minds and loyalties of the millions in the uncommitted and underdeveloped areas of the world, it is suggested that the Government contribute funds to existing colleges and universities to help train United States technical assistance representatives for service abroad.

(9) *Marshaling free world's scientific resources.*—Inasmuch as we are seriously outnumbered in population by the Russians, co-operation of the free world countries is proposed to marshal scientific resources and to develop common programs. Such co-operative programs might include scholarships and fellowships, equipment and libraries, summer institutes, and visiting professors. Co-operation with underdeveloped countries would be emphasized. Some organization such as NATO in Europe or SEATO in the Pacific might be utilized as the co-ordinating body.

(10) *Atomic energy: Expansion of reactor training programs.*—The estimated need for reactor technicians is 2,000 for each of the next 3 years. We are at present producing only 500 per year. Moreover, within the next 15 to 20 years there is expected to be some fifty to sixty million kilowatts of electric generating capacity in nuclear powerplants, with a resulting large increase in demand for reactor technicians. It is therefore recommended that all possible steps be taken to increase the number of such technicians, including:

1. expansion of the Government's reactor training programs;
2. increasing the number of "reactor kits" containing subcritical assemblies to be made available to universities for instruction in reactor technology; and
3. providing aid and assistance to universities in the construction and operation of research reactors.

Provisions incorporating the recommendations in this section are contained in proposed legislation recently reported out to the Congress by the Joint Committee.

Conclusion

All recommendations and conclusions lead back to the basic fact that there is no single or final solution to the shortage of skilled manpower and that the efforts of all, Government and private, will be required if we are to make progress. As the shortage of scientific and engineering manpower is only part of a larger national problem involving the need for competent trained people in all fields of endeavor, our main objective should be the development of all our intellectual resources.

The greatest need at the present time is for a vigorous and coherent national policy to provide guidance for all participants, Government and private, in a co-ordinated program of remedial action. In framing such a program it is important to bear in mind that present shortages of manpower in this country are a symbol of success, not of failure; that inherent in our constant growth and explansion are increasing demands for skilled persons of all kinds—demands which may never be completely met. This is not a crisis to be deplored. It is an unprecedented achievement and eloquent testimony to the suc-

cess of a social experiment which, when this Republic was founded, was only a dream and an ideal.

Now that so much of the dream has come true, the continued approach to the ideal can only be maintained by the full development and the wise use of all the brainpower of all the men and women who are part of the experiment. This alone is the problem of the future.

JOINT COMMITTEE ON ATOMIC ENERGY

CLINTON P. ANDERSON, *New Mexico, Chairman*
CARL T. DURHAM, *North Carolina, Vice Chairman*

RICHARD B. RUSSELL, *Georgia*
JOHN O. PASTORE, *Rhode Island*
ALBERT GORE, *Tennessee*
HENRY M. JACKSON, *Washington*
BOURKE B. HICKENLOOPER, *Iowa*
EUGENE D. MILLIKIN, *Colorado*
WILLIAM F. KNOWLAND, *California*
JOHN W. BRICKER, *Ohio*

CHET HOLIFIELD, *California*
MELVIN PRICE, *Illinois*
PAUL J. KILDAY, *Texas*
JOHN J. DEMPSEY, *New Mexico*
W. STERLING COLE, *New York*
CARL HINSHAW, *California*
JAMES E. VAN ZANDT, *Pennsylvania*
JAMES T. PATTERSON, *Connecticut*
JAMES T. RAMEY, *Executive Director*

SUBCOMMITTEE ON RESEARCH AND DEVELOPMENT

MELVIN PRICE, *Illinois, Chairman*
CARL T. DURHAM, *North Carolina*
CARL HINSHAW, *California*
JAMES E. VAN ZANDT, *Pennsylvania*

JOHN O. PASTORE, *Rhode Island*
ALBERT GORE, *Tennessee*
BOURKE B. HICKENLOOPER, *Iowa*
GEORGE E. BROWN, JR., *Staff Member*

. . . developments clearly indicate that the next few years will see the emergence of a college-preparatory curriculum in mathematics which is more advanced, better organized, and more modern than the traditional program now in effect. "Traditional program," as used here, denotes a program of four semesters of algebra, two of plane geometry, one of solid geometry, and one of trigonometry.

The new program will include some analytic geometry and a substantial introduction to calculus, subjects which have seldom been presented in secondary schools in the United States.—*Taken from "Mathematics Tomorrow," by Frank B. Allen, Chairman, Mathematics Department, Lyons Township High School, LaGrange, Illinois.* NEA Journal, *May 1957.*

The changing concept of trigonometry as a school subject[1]

HERMAN ROSENBERG, *New York University, New York, New York.*

A résumé of the status of trigonometry and some comments on desirable changes are presented.

Introduction: twentieth-century upsurge in interest in trigonometry

A HALF-CENTURY has elapsed since Robert E. Moritz complained in an article in *School Science and Mathematics* of a lack in the United States in the year 1908 of even a single published article dealing exclusively with the teaching of trigonometry. Since then, evidence of a marked increase in attention to the problems of teaching trigonometry has been provided by the appearance in the literature of numerous references to such problems—not only in relatively brief articles, but also in extended treatments, such as books and dissertations.

Certainly, one factor in stimulating such interest in the teaching of trigonometry may be found in a recent enrollment report of the United States Department of Health, Education, and Welfare. The report states that since the year 1900 the high school enrollment of trigonometry students in the United States has increased fifteen times. The need, too, for such attention to trigonometry arises from the facts that relatively "complete" courses in trigonometry are offered in both senior high schools and colleges and that topics from trigonometry are taught on three educational levels:

1. The junior high school level in courses in elementary algebra and "general" mathematics.
2. The senior high school level in courses in plane geometry, algebra, and "general" mathematics.
3. The college level in "general" mathematics courses for college freshmen.

At the same time, some of the most powerful stimulation of the recent activity in the area of trigonometry may be traced to determined efforts to meet more effectively the mathematical needs of gifted students, particularly in the light of the following factors:

1. Recent disturbing reports of increasingly more critical shortages of competent engineers, scientists, and other personnel trained for the mathematical leadership needed for the continued growth of the American economy and for possible protective purposes in a time of a bitter "cold war" between the United States and Soviet Russia.
2. Suggestions that special students, such as engineering students, be provided with a more thorough knowledge of trigonometry and be introduced to concepts of higher mathematics earlier in the educational program. Such suggestions have arisen partly because of beliefs that " . . . the mathematical maturity of a student depends to some extent on the length of time he has been familiar with and used a concept"[2] and

[1] An address delivered before the Research Section of the Annual Meeting of the National Council of Teachers of Mathematics at Philadelphia, Pennsylvania, March 28, 1957.

[2] James H. Zant, "A Program for Determining the Mathematical Needs of Engineering Students," THE MATHEMATICS TEACHER, XLIII (March 1950), 93–94.

partly because of a desire to reduce the precious time now required for mastery of the mathematics needed in a world which continues to devour more and more mathematics.

3. The existence of a tremendous *lag* between the phenomenal growth of mathematics in modern times and the potential utilization of certain aspects of this expansion in the teaching of trigonometry.

The rise and fall of "static" trigonometry

In view of the essential role of trigonometry in modern civilization, it is pertinent to ascertain the present conception of trigonometry as a school subject. Since it has recently been reported that all students in Russian schools are required to study some trigonometry, it might be expected that our American schools are meeting the challenge by offering truly worth-while trigonometry courses. However, one reads with considerable dismay Dr. Fehr's recent description of present-day American trigonometry: "... for two months the student climaxes his mathematics study with tedious and no longer used logarithmic solution of oblique triangles."[2]

To gain some insight into the nature and possible solution of the present American problem, it may be instructive to analyze briefly certain basic steps in the historic evolution of trigonometry. It may be recalled that in the second century B.C. the so-called "father" of trigonometry, the Greek mathematician Hipparchus of Rhodes, regarded his "child" as a mere adjunct to his astronomical endeavors. In the medieval period, Moslem mathematicians developed the Indian trigonometry and continued the ancient interest in trigonometry for its uses in astronomy. At the same time, European mathematicians such as Leonardo of Pisa (Fibonacci of "golden section" fame) became familiar enough with Arabic trigonometry to apply it to surveying considerably beyond the primitive forms of ancient Egypt. It is significant to observe that:

1. The purposeful divorce of trigonometry from astronomy (and its consequent emergence as a distinct, independent subject)—usually assigned to the German mathematician Johann Müller ("Regiomontanus") in his treatise of 1464, *De triangulis omnimodis*—took place about sixteen centuries after the time of Hipparchus.
2. It took another three centuries after this "escape" of trigonometry from astronomy before trigonometry was finally "captured" by mathematical analysis under the leadership of such prolific eighteenth-century mathematicians as Euler.

A careful study of the history of trigonometry reveals that:

1. Changes in trigonometry have emerged rather slowly, thus contributing to the generally slow progress over the years in improving the teaching of trigonometry in the schools.
2. Throughout most of its long history, trigonometry was considered a mere "servant" of surveying and astronomy, thus contributing to the present all too frequent tendency in the schools to conceive of trigonometry rather narrowly as only the study of "triangle measurement."

Now, such a teaching emphasis on the solution of triangles in trigonometry courses might be easily justified in the pre-modern period when astronomy was the major science and the only science requiring any considerable use of mathematics (primarily the solution of triangles). However, such emphasis is hardly justified in our modern era, for:

1. Positional astronomy has "... receded to a subordinate routine in modern astronomy," and "the trigonometric func-

[2] Howard F. Fehr, "Current Problems in Teaching Secondary School Mathematics," *The New Jersey Mathematics Teacher*, XIII (October 1956), 13.

tions—for reasons in no way connected with the solution of triangles—have become the indispensable mathematical aid from celestial mechanics to spectroscopy."[4]

2. In the modern sciences, astronomy has "... become but one science of many, some being of perhaps even greater practical importance than astronomy in a scientific civilization. Here, again, ... the trigonometric functions ... were to prove indispensable, and again for no reason even remotely concerned with the solution of triangles."[5]

The fact is that one of the primary reasons for the modern scientific importance of the sine and cosine functions is that they possess the extremely useful property of being periodic functions. Thus, a trigonometry course reasonably in tune with our times considers applications of trigonometry to such periodic phenomena as music, light, and electricity, and, therefore, includes a meaningful discussion of such fundamental elements as amplitude, period, and frequency. Today, trigonometry courses which are guilty of the outmoded excessive emphasis on the solution of triangles and the corresponding excessive neglect of the graphic aspects of trigonometry lend further weight to such recent charges as the following:

1. Secondary school mathematics "... has become, and is continuing to become, less and less relevant to modern mathematics and its applications to modern science."[6]
2. That "... by and large the mathematics curriculum in general use today looks remarkably like the 1890 model" and "... most students who stick with mathematics throughout the four years of high school follow the same ruts their grandfathers followed before them. The high school curriculum today shows few, if any, signs of the important developments that have taken place in mathematical science since the seventeenth century."[7]

Perhaps, at this point, it may be pertinent to comment briefly on the present place of spherical trigonometry in the secondary school curriculum. Spherical trigonometry is older than plane trigonometry. Indeed, due to the ancient and medieval emphasis on astronomy, plane trigonometry was formerly quite subordinate to spherical trigonometry. However, few of today's schools would contemplate offering full courses in spherical trigonometry on the secondary-school level, although topics from the subject are sometimes introduced in courses such as plane trigonometry, solid geometry, or "general" courses in senior high school mathematics.

The explanation for the decline of spherical trigonometry has been rather vigorously stated by Bell in these words: "Spherical trigonometry is no longer in the living stream of mathematics .... Unless a student today requires the subject for some definite routine, such as the old-fashioned positional astronomy, he need not even know that spherical trigonometry exists. Of all the subjects in elementary mathematics, spherical trigonometry is probably the deadest and the most repulsive to anyone with the faintest stirrings of a feeling for vital mathematics."[8]

Fortunately, some undesirable practices in the teaching of trigonometry have been disappearing. Thus, for example, trigonometry courses have happily been emerging from those "good old" school days of Felix Klein in the nineteenth century, when seven-place logarithm tables were employed in schools, "... the reason as-

[4] Eric Temple Bell, *The Development of Mathematics* (New York: McGraw-Hill Book Co., 1945), p. 105.

[5] *Loc. cit.*

[6] E. P. Northrop, "Modern Mathematics and the Secondary School Curriculum," The Mathematics Teacher, XLVIII (October 1955), 386.

[7] Henry S. Dyer *et al.*, *Problems in Mathematical Education* (Princeton: Educational Testing Service, 1956), pp. 19, 22, 23.

[8] *Op. cit.*, p. 105.

signed being that the pupils would obtain in this way an impression of the 'majesty' of numbers."[9] Again, in the teaching of proofs of trigonometric identities, the modern spotlight of reason has been thrown on the outmoded "tradition" which regards working with both sides of a trigonometric identity as "sinful."

The challenge for tomorrow: the creation of "dynamic" trigonometry

Essentially, then, the trigonometry of today's schools is largely a "static" trigonometry, very much in need of transformation into a "dynamic" trigonometry which would be responsive to modern developments in science, mathematics, and education. What fundamental changes are needed to make such a transformation possible?

For one thing, trigonometry courses should banish to oblivion the fallacious view that there is little opportunity in such courses for the exercise of logic. Such a view arises in some quarters from an unfortunate emphasis on the routine solution of triangles. On the contrary, it should be realized that the power of trigonometry, as in all of mathematics, lies in the method of development: deductive logic. Thus, the structure of trigonometry is built by first carefully defining such terms as the tangent function, inverse sine function, and radian. Deductive reasoning is then applied to such definitions and to previously accepted assumptions and previously proved theorems of geometry and algebra in order to establish the major properties of the functions studied in trigonometry, properties such as the reduction formulas, double angle formulas, law of cosines, etc.

Furthermore, analysis of the nature of trigonometry reveals that trigonometry has three basic components:

1. The analytic component: Trigonometry may be conceived as a part of mathematical analysis, for trigonometry deals with the properties of certain transcendental functions.
2. The geometric component: Trigonometry may be conceived as a part of geometry, for trigonometry is concerned not only with the applications to certain geometric configurations of various relationships existing between the trigonometric functions but also with the graphing of such functions.
3. The algebraic component: Trigonometry may be conceived as a part of algebra, for trigonometry utilizes algebraic concepts (such as the formula, equation, and polynomial) as tools in the development of the analytic and geometric components of the subject.

At the same time, modern mathematics has frequently been divided into the major branches of analysis, geometry, and algebra, although, as Weyl has indicated, " . . . the unity in diversity in modern mathematics makes a clear-cut division . . . practically impossible."[10] As Northrop states the matter, " . . . the concepts and methods [in modern mathematics] that in our days as students were neatly compartmentalized into algebra, geometry, and analysis have now become mixed beyond all recognition."[11]

Now, such an analysis of the basic components of trigonometry and of higher mathematics unveils the potential role of trigonometry as an "integrating agent" in mathematical instruction, for trigonometry stands at the "crossroads." On the one hand, it utilizes essential secondary-school mathematics, and, on the other hand, it is in a strategic position to open up avenues to branches of higher mathematics. These considerations provide support for such current curricular experimentation with nontraditional secondary-school courses in "integrated" mathematics as the work of the University of Illinois Committee on

[9] Felix Klein, *Elementary Mathematics from an Advanced Standpoint* (New York: Dover Publications, 1945), I, 174.

[10] Hermann Weyl, "A Half-Century of Mathematics," *American Mathematical Monthly*, LVIII (October 1951), 552.

[11] *Op. cit.*, p. 387.

Secondary-School Mathematics and the recent effort to integrate trigonometry with intermediate algebra in the New York State course of study for eleventh-year mathematics. Encouragement for the destruction of the high walls isolating algebra, geometry, and trigonometry has come from persons such as Carroll V. Newsom, who, a decade ago, indicated his view that trigonometry " . . . can be taught in such a manner that concepts in both algebra and geometry receive clarification . . . . Modern attempts at unification certainly deserve every encouragement."[12]

But the creation of a "dynamic" trigonometry requires more than the integration of trigonometry with traditional courses or topics in secondary-school mathematics. A truly modern, "dynamic" trigonometry should attempt to utilize concepts of modern mathematics. Hence, a major problem in developing such a course will be that of locating and selecting modern concepts that are both fruitful and suitable. The importance of functions in the analytic component of trigonometry indicates that some concepts may be selected from the area of modern mathematics known as mathematical analysis. Thus, it may be helpful to explore the possibilities of utilizing from the functions of a complex variable such concepts as the polar and exponential forms of complex variables to derive, for example, the law of cosines. Similarly, the geometric and algebraic components of trigonometry suggest that appropriate alternative procedures and elements of insight may be discovered in such areas of modern mathematics as higher geometry (particularly analytic geometry and modern Euclidean and non-Euclidean geometry) and higher algebra (both classical college algebra and modern abstract algebra). The modern area of vector analysis is inviting to teachers of modern trigonometry since the basic concepts of vector multiplication are defined in terms of elementary trigonometric functions. Again, more doors may be opened by the possibilities of allying the modern congruence concepts of the theory of number with the fundamental periodic properties of trigonometric functions. And, of course, since the area of the modern foundations of mathematics includes some intriguing relations between mathematics and logic, it beckons the trigonometry teacher interested in providing students with greater insight into theories of proof peculiar to trigonometry. Indeed, in the light of numerous possible connections between trigonometry and major areas of modern mathematics, it may well be that future research will reveal that trigonometry is the ideal nucleus around which to introduce key concepts of modern mathematics into the mathematics curriculum of the secondary school.

What, then, are some major characteristics of a genuine "dynamic" trigonometry for tomorrow's schools? From what has been said, such a course cannot be just a dull, routine study of the solution of triangles and other polygons. Instead, it should be conceived as a broad, general area of mathematics in which the geometric point of view is only one of many possible interpretations. Thus, the variable x in such functions as $\sin x$ and $\cos x$ need not be restricted in interpretation to represent only an angle, but rather may be conceived in the more modern and useful manner as a generalized variable which may be an abstract number (real or complex), time, or some other magnitude. Unfortunately, as Newson and Randolph have reported, most trigonometry textbooks have regarded trigonometry as " . . . fundamentally concerned with angles. In fact, the idea is presented with such insistence and emphasis that advanced instruction in science and mathematics is definitely hampered."[13] Furthermore, a genuinely modern trigonometry course destroys the

[12] Carroll V. Newsom, "A Philosophy for the Mathematics Teacher," THE MATHEMATICS TEACHER, XL (May 1947), 195–196.

[13] C. V. Newsom and J. E. Randolph, "Trigonometry Without Angles," THE MATHEMATICS TEACHER, XXXIX (February 1946), 66.

fallacy that the only "true" trigonometry is the trigonometry based on the assumptions of Euclidean geometry. The truly modern trigonometry course explodes the myth that the only type of trigonometry worthy of study is the trigonometry of the circular functions, for the modern course may be regarded as an introductory exploration of the theory of the elementary transcendental functions: trigonometric, inverse trigonometric, logarithmic, exponential, hyperbolic, etc. It is a course in which stress is placed on the modern applications of the transcendental functions not only to the natural sciences but also to the social sciences. It is an "emancipating" course in which (with the possible assistance of historical materials and/or inductive methods) students begin to make the exciting discoveries that many alternative sequences are possible in trigonometry, that many alternative methods are available from mathematical logic for proving trigonometric identities even within a given sequence of content, and that there do exist "other worlds" in which, for example, the sine of an angle may be greater than one or in which negative numbers may have logarithms. It is a warm, uplifting course in which the development of trigonometric skills is supplemented with generous portions of mathematical insight and power, in which formal deductive procedures are enriched by informal laboratory techniques, in which a premium is placed not only on trigonometric knowledge but also on mathematical appreciation and perspective, and in which the philosophy of respect for individual human differences is paramount.

Concluding remarks: breaking through the trigonometric barriers

The program of modernizing our trigonometry courses will not, of course, be without difficulties and/or pitfalls for both trigonometry students and trigonometry teachers. In considering the needs of trigonometry students, we should be wary of the fallacious view that "if it's modern, it must be good." It is conceivable that, for some students, certain aspects of modernized trigonometry may be too difficult and/or less useful than other topics. Thus, the wise procedure may be to select concepts of modern mathematics with great discrimination, to introduce concepts of modern trigonometry into the secondary school gradually, to avoid excessive abstractness, and, perhaps most of all, to present such material in highly meaningful fashion. For, as the "meaning theory" of mathematical instruction may "push" mathematical topics from the elementary school *up* into the secondary school, and as current emphases on modern mathematics may "push" topics from college mathematics *down* into the secondary school, the pressure on the secondary school may increase, and the problem of the articulation of, say, secondary-school trigonometry with college mathematics may become even more acute. Under such conditions, secondary-school mathematics could easily degenerate into a superficial coverage of topics—unless an over-all, organized effort is made to delete obsolete mathematical topics and refashion the rest of the secondary-school mathematical program in a truly meaningful manner. Some valuable aid in this direction may come from such current groups as the Curriculum Committee of The National Council of Teachers of Mathematics and the College Entrance Examination Board Commission on Mathematics.

In considering the needs of trigonometry teachers, it should be realized that their enthusiastic acceptance of a program of modernized trigonometry will depend on the gradual elimination of at least two major barriers:

1. Lack of extensive knowledge of modern mathematics. In this connection, such efforts as the new *Insights into Modern Mathematics*[14] should prove to be a

[14] *23rd Yearbook*, The National Council of Teachers of Mathematics, Washington, D.C., 1957.

boon in clarifying for teachers concepts of modern mathematics relevant to elementary trigonometry.

2. Resistance to curricular change. The need for reducing such resistance makes it extremely urgent that future articles, books, college courses, institutes, etc., spell out in highly vivid detail the implications of these concepts of modern mathematics for trigonometry teachers. This activity should facilitate the trigonometry teacher's effective presentation of such implications on levels suitable to the interests and maturity of our rapidly growing secondary-school population.

In conclusion, these barriers to the creation of a "dynamic" trigonometry must not be and, indeed, are not insurmountable, for our conquest of them through cooperative planning and research will bring us closer to a living mathematics curriculum worthy of our evolving American democracy.

Have you read?

HITLAND, MELVIN, and GLENN, HAROLD. "A Program for the Mathematically Gifted," *California Journal of Secondary Education*, October 1957, pp. 334–337.

The article tells the story of a program carried out in Long Beach, California. Algebra and plane geometry were telescoped into a one-year program, and gifted students only were enrolled. The gifted were chosen from students of high standing on mathematics tests, an I.Q. of 110, and a recommendation from their arithmetic teacher.

It was found among other things that these students did better: they learned how to study, those not specializing in mathematics advanced in other areas, and many of them continued in advanced mathematics.

You will be interested in the procedure by which pupils simultaneously used algebra and geometry texts and were required to do one hour of homework each day. This will give you an idea of what might be done in your school.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

STONE, M. H. "Mathematics and the Future of Science," *Bulletin of the American Mathematical Society*, March 1957, pp. 61–76.

This is a copy of the annual Gibbs Lecture of the American Mathematical Society, presented by Dr. Stone in 1956. It is an excellent illustration of the value in maintaining a balance between applied and theoretical mathematics. Dr. Stone's simple statement of the connection between science and mathematics points out, "Science is reasoning; reasoning is mathematics, and therefore, science is mathematics."

The author defines logic and science very well. You will be interested to note his analogy between better mousetraps and what is happening in mathematics today, and in its evaluation. Dr. Stone believes a reappraisal is necessary in the light of changes in mathematics needed for the new knowledges of physical conditions. This may show the way for more unification within the subject itself, as well as with other disciplines.

You will be interested in one of Dr. Stone's last comments that in secondary mathematics the scale has been heavily weighted on the utilitarian side. This should be rethought. I think you will profit from considering this article.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

ZANT, JAMES H. "Freshman Mathematics for Engineers," *Journal of Engineering Education*, May 1957, pp. 739–744.

Little that is taught in mathematics at present makes a significant contribution to the needs of engineering students, says this author. For more significant and efficient use of mathematics, the student should be introduced to the axiomatic method, logic, sets, number system, matrices, group theory, vector space, and modern algebra.

To do this, it would appear that four major modifications would be necessary. First, eliminate much of the compartmentalization; second, remove the traditional material that has proved nonessential; third, introduce modern mathematics earlier; and fourth, introduce more mathematics in high school by selection of the best material.

Such a program is objected to by the traditionalist. It is hampered by lack of sufficiently well-prepared faculty. Some persons feel that the skills will be slighted, but it is recommended these be maintained through new courses like statistics and quality control. A very encouraging factor would be more communication and co-operation between engineers, engineering faculties, and instructors of mathematics. I think this article has many implications for those of us teaching high school mathematics.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Let's teach statistics

CLARENCE E. OLANDER, *Senior High School, St. Louis Park, Minnesota.*

Statistics is being recognized as a suitable course for high schools. Here is one high school's answer to the demand for more statistics in the high school program.

THE SEVENTEENTH SUMMER MEETING of the National Council of Teachers of Mathematics included seminar sections on the structure of mathematics, sets and Boolean algebra, finite geometry, game theory, and statistics. In each of these sections a recognized authority lectured on the topic. In many cases implications for secondary-school mathematics were included in the lectures. Other sections of the meeting revolved around the problem of teaching these topics and involved high school mathematics teachers. It is both encouraging and comforting to know that pure mathematicians and educators are pooling their efforts to bring secondary-school mathematics within the framework of modern ideas and concepts.

Meetings such as the one mentioned above, as well as articles and speeches written and delivered by prominent authorities, attest to the fact that the revolution taking place in mathematics is beginning to affect the high school curriculum. Much is being done to revitalize secondary-school mathematics. The very foundations of traditional, or classical, mathematics are being subjected to critical examination. Experimental programs have been organized which have either discarded or modified some of the content of traditional mathematics. New vocabulary and more primitive notions emphasizing the structure of mathematics have been introduced. Several national, state, and local committees are studying the mathematics curriculum, and certain recommendations have already been made.

One such recommendation concerns the teaching of statistics to secondary-school students. A knowledge of statistical reasoning is necessary to appreciate and understand many of the activities of man within our modern society. In most cases, however, the only statistics included in current mathematics programs is a simple descriptive (primarily graphic) presentation.

It is the purpose of this paper to discuss the why, where, what, and how concerning the teaching of statistics in secondary schools. The writer is a high school mathematics teacher and recently has completed a unit on statistics designed for high school students. The ideas developed in the unit were taught to a class of high school sophomores. This class had a median I.Q. of 125 with a range from 119–136. A class of this kind is not too difficult to obtain in moderately large high schools where a philosphy of homogeneous grouping for mathematics instruction prevails. Although most of the ideas discussed in this article have been developed for the superior student in senior high school mathematics, many of the concepts of statistics are appropriate for all secondary-school mathematics students.

It is not difficult to justify the teaching of statistics to mid-twentieth-century high school students. With increasing frequency the average citizen is asked to "digest"

information about a large set of elements obtained from a sample of this set. To interpret with intelligence statements involving quantitative concepts, one must have a knowledge of the vocabulary and methodology of statistics. Secondary students should be aware of the increasingly important role that statistical reasoning is playing in our society. Applications of statistics to such areas as safety, insurance, opinion polls, sports, consumer preference, cost of living, and scientific experimentation are becoming more prevalent each day. Furthermore, in the teaching of statistics the teacher has a beautiful opportunity to compare the various kinds of reasoning, and specifically to contrast statistical thinking with the postulational thinking of formal mathematics.

If the teaching of statistics is to be formalized and become a part of the secondary-school curriculum, general objectives such as the following must guide the development of a sound program in statistics:

1. Understanding of the types of problems in our society that can be solved through the use of statistics.
2. Understanding of the role of mathematics and probability in statistical reasoning.
3. Understanding of basic statistical techniques, concepts, and methods.
4. Ability to define a problem, design a simple experiment, and carry out the proper statistical analysis necessary to draw valid conclusions.
5. Appreciation of the limitations of statistical inference and an understanding of the generalizing ability of experimental evidence.

Let's accept the assumption concerning the "why" of teaching statistics and consider some problems concerning the "where" of teaching statistics.

Statistics includes a body of knowledge which could be broken up into facts, skills, principles, concepts, techniques, and methodology. As such, a program in statistics could be developed to parallel that of any other academic subject, and statistics probably could be taught as a separate subject beginning in the elementary school and extending throughout the secondary school. It is beyond the scope of this paper to discuss statistics from this point of view.

If statistics is not going to be taught as a separate subject, then a program in statistics must be developed within the limits of some other academic area in the curriculum. The most logical place for this is mathematics. Although mathematics and statistics are not identical—mathematics is a deductive science, while statistics attempts to evaluate the outcomes of scientific experimentation—the role of mathematics in statistical reasoning is sufficiently apparent to justify including statistics in the mathematics curriculum.

Up to this point we have assumed that statistics should be taught to secondary-school students and that statistics should be a part of the mathematics curriculum. If statistics is going to be included in the mathematics curriculum, we are faced with the problem of implementation involving grade level and subject matter. Actually at this stage there are two alternatives. A continuous sequential program in statistics can be developed to be included in the mathematics program for grades seven through twelve. This would mean that all students would receive some training in statistics. The level of difficulty of the ideas and concepts would progress in the same way that concepts in mathematics are developed over a period of years. The other possibility is to develop a course in mathematical statistics to replace a semester of traditional mathematics. This would probably be an elective course for advanced mathematics students. Regardless of the alternative selected, the development of a program in statistics must be planned and continuous. It must not be done incidentally!

Since statistics is a relative newcomer that has not yet been accepted by some mathematicians and educators, it is prob-

ably unwise to make a decision concerning the alternatives mentioned above. Statistical thinking at the present time is in its initial stage of exploration as far as secondary students are concerned. Teachers must experiment with various topics and approaches to establish assumptions and theories which will underlie the teaching of statistics in the high school.

The next problem is to discuss the "what" concerning the teaching of statistics. An attempt will be made to suggest topics that will provide secondary students with some reference and background material for both the content and spirit of statistical reasoning. These topics could be developed over a number of years or serve as the basis for a "unit," or course, in statistics.

To achieve the objectives set forth in this paper, appropriate content and experiences must be provided for the students. Mathematics is used throughout the unit as an applied science. It is a tool that is applied to the data to enable certain statistical techniques to discover new knowledge. However, the collection of these facts into broader principles and generalizations requires a kind of reasoning quite different from the deductive reasoning of pure mathematics. Nevertheless, mathematics is used again to give an expression of the margin of uncertainty concerning the generalizations or inferences that are made. Throughout the study of statistics, the different roles that mathematics plays in statistical reasoning should be emphasized.

In most treatments of statistics a distinction is made between descriptive and inferential statistics. Descriptive statistics usually refers to the tabulating, graphing, and summarizing of data by means of measures of central tendency and variability. Inferential statistics concerns the drawing of conclusions about a population from data collected from a sample of this population. This second aspect of statistics is based on probability, a study of chance or random variation. But rather than emphasize this dichotomy, it is better to talk of the process of statistical reasoning as making use of descriptive statistics in carrying out the process of statistical inference. Statistical reasoning involves both of these aspects and as such is considered a part of the general scientific method.

Consequently, in the development of a program in statistics and probability the students should first obtain a feeling for the nature of statistics. This can be accomplished by discussing the various aspects of statistical reasoning as applied to an introductory problem. This would include a definition of the problem, obtaining the sample, statement of the hypothesis, and procedures involved in testing the hypothesis. An introduction to sampling that involves a discussion of sampling designs and good and bad sampling techniques should follow. Some work on descriptive statistics involving frequency distributions, averages, and the standard deviation should be included.

Next, considerable time should be spent studying probability. Such concepts and ideas as permutations and combinations, the binomial theorem, probability theorems, empirical probability, and probability distributions should be emphasized. In studying probability distributions, the student should become familiar with the binomial and normal particular random variables and with the sampling distributions for sample proportions and sample means. The student is now ready for a thorough study of statistical inference involving statistical estimation and testing statistical hypotheses. A theoretical discussion of statistical reasoning should complete the study. This should involve a discussion of the logical role of statistics, limitations of statistical reasoning, and a comparison of kinds of reasoning.

No doubt there are those who will feel that the elements of inferential statistics are too difficult for high school students. Obviously it is true that many of the concepts involved cannot be adequately explained at the high school level. This

would require a substantial background in the mathematical theory of statistics. However, on the basis of evidence obtained from high school students, the writer firmly believes that certain fundamental concepts and methodology, as well as some elementary but useful techniques, can be acquired by secondary students. Much experimentation is needed to establish a curriculum in statistics for secondary-school students.

It is quite difficult to separate the "what" and "how" in a particular teaching situation. Let's consider some pertitent "how" problems before returning to a more specific discussion concerning the content.

As in all effective teaching, the methods used in teaching statistics should be based on scientific research and sound principles from the psychology of learning. Appropriate learning activities and materials of instruction must be used to fit the proper teaching situation. One suggestion concerning individual differences is to organize committees. For example, less able students could be responsible for the collection and presentation of data, while the more able students could be responsible for the design of the experiment, the analysis of the data, and the inferences from the data. This would give all students a chance to make a real contribution to the solution of a problem. But it is not our purpose to include further specific teaching suggestions in this article. Let us turn immediately to another very important problem.

What should be the approach in teaching statistics to secondary-school students? The approach must be such that the students will not "lose sight of the forest because of the trees." The major objective is not to create research statisticians but to give students some impression of statistical reasoning and the kinds of problems that can be solved through statistical reasoning. Also, the student must be aware of the limitations of such "solutions."

The approach must allow students to develop an understanding of the statistical methods under consideration. The fact that a student can perform certain mathematical computations regarding the use of statistical techniques indicates very little as to understanding. Furthermore, a knowledge of the steps involved in testing a hypothesis does not guarantee that the student has a fundamental knowledge concerning the acts of inference. Oftentimes the blind, mechanical use of various formulas seems to add to the general confusion. The interpretation of statistical reasoning can be understood most easily if the general problem is first studied from its logical, rather than its mathematical, aspects.

This approach must be reflected in the content and teaching methods. Furthermore, any techniques that are used are considered within the broader framework of statistical reasoning. Although it is impossible to discuss all of the content completely, let's now consider some specific ideas that might well receive attention in the secondary schools.

If our approach is going to emphasize understanding, then the student from the very beginning must have some idea of the meaning of statistical reasoning. Too often statistics means nothing to the student in addition to numerical data. This is the case even after students have studied statistics. The student should understand that statistical reasoning begins with a clear and precise definition of a problem and ends with probability statements concerning the hypothesis formulated from the problem. Statistics can be defined as a science which is concerned with acquiring and analyzing data obtained for some specific purpose. The data is obtained according to definite rules set up in advance and, as such, statistics must be associated with an experiment. But to conduct an experiment we must have a problem to which the experimental method can be applied. Students can begin to appreciate that these problems grow out of critical observation

of actual conditions that exist within some field of knowledge. Experimentation arises out of the intellectual desire to gain control over these problem situations. After an experiment has been carried out, statistics enables us to draw conclusions about populations after investigating samples of these populations. Consequently, before the student begins to worry about means, standard deviations, and probability distributions, he has some notion of statistical reasoning. He knows that essentially he must be concerned with the definition of the problem, design of the experiment, analysis of the data, and inferences about the population. He doesn't know how to analyze the data as yet. But he knows why he has collected the data and that there are certain techniques which will enable him to analyze his data.

In teaching statistics to secondary students, the first few periods should be spent in acquainting the student with the aspects of statistical reasoning just mentioned. This can be done by discussing a simple experimental design in which the objective is to estimate the effects of differential treatments. The importance of making observations under conditions which the experimenter can control should be emphasized at this time. Also the role played by the control group should be discussed as a necessary prerequisite of good experimentation. It is through the use of controls that results of experiments become comparative.

In the discussion of the sample experiment the student will appreciate the importance of obtaining a random sample. If the sample is not a random sample, the statistical tests that are used in analyzing the data will not be of any use. In simple random sampling any element in the population has the same chance or probability of being selected as any other. This enables the experimenter to make quantitative statements with known probability of error from sample to population.

This brings up the notion of probability. Students cannot study statistical inference without a knowledge of probability. They must realize that the mathematics upon which statistical reasoning is based is a kind of mathematics that implies on the average certain events may be expected. Probability can be introduced by discussing questions whose answers are stated in the language of probability. Probability can be described as a measure of the faith that can be placed in conclusions made about a population from the data of a sample of that population. The importance of probability cannot be overestimated. In statistical inference, probability theory is basic in the way the sample is selected from the population and in the way the estimates are prepared from the sample. Consequently, considerable time must be spent in studying probability. For many students it will be necessary to teach some new algebraic skills at this time.

Let's return to our example experiment which we are using to introduce statistical reasoning. Assume that we have computed a mean for two comparative groups. One mean is slightly greater than the other. We inquire of our students if the experimenter should claim on the basis of this evidence that one treatment is superior to the other. This brings up the crucial question concerning the likelihood, or probability, of getting this obtained difference between the means. What is needed to settle the argument, then, is an indication of the likelihood, or probability, of getting this obtained difference between the sample means if there is no difference in the effectiveness of the two treatments. If it can be shown that this probability is extremely small, then the experimenter is in a position to say that something more than chance factors must have been operating to produce the difference. At this time the students should understand and appreciate that the purpose of several statistical techniques is to provide this probability figure. This probability figure refers to the probability of getting the obtained results if there were no difference in the treatment of the samples. Following this discussion,

it is appropriate to mention the meaning of a null hypothesis and test of significance. Also students should have an understanding of what is meant by accepting and rejecting a hypothesis. They should understand that the hypothesis is a statement concerning the population, but that the decision concerning the hypothesis is based on results obtained from a sample of this population.

During this introduction to statistical reasoning, it is appropriate to define some of the terms that will be used throughout the work on statistics. These terms could include elementary sampling unit, population, universe, target population, sample population, sample, precision, accuracy, and random sample. These terms are necessary in discussions of sampling surveys as well as in discussions of comparative experiments.

The student now has an overview of the general procedure. The various mathematical aspects and specific statistical techniques necessary in investigating a problem should be explored at this time. Students should appreciate that techniques are a part of methodology, but that techniques arrive at facts, while methodology arrives at principles. The different techniques and concepts that are studied throughout the unit will take on added meaning and significance if they are related to some problem situation.

If inferential statistics is given primary consideration in secondary mathematics, it is unnecessary to spend a great deal of time on the various descriptive aspects of statistics. Some time should be spent computing means and standard deviations for small sets of data. Students should be introduced to the conventional summation notation at this time. If the sets of data that are used are small, it is not necessary to spend a great deal of time summarizing data by means of class intervals. Some teachers may wish to digress and include a great deal of practice involving tabulation of data and computing the percentiles. This seems unnecessary and usually "loses" the students in arithmetic computations.

It is certainly true that a practicing statistician works with numerically large sets of data. But the concepts of inferential statistics are quite abstract, and a first exposure for high school students should involve small sets. Concerning the data, however, it is desirable that the students have some notion of the role of number and how it is used to quantify data. Data should be classified as discrete or continuous, depending on how quantification is achieved. This is important, since in studying probability distributions a variable is defined as an observation which can assume different values. The variable is discrete if it takes on a set of values such as 0, 1, 2, . . . , n. However, for the normal distribution the random variable is a continuous variable in the range from ($-\infty$ to ∞). In using the binomial distribution, we discuss the probability that our random variable has some particular value, but for the normal distribution we discuss the probability that it is in some range.

After the students have an understanding of sampling, it is convenient to discuss sampling distributions. This must be preceded, however, by considerable work in probability. The work in probability should approximately follow the suggestions included in this article. The students should have a knowledge of the binomial theorem. This will enable them to compute the probability of exactly x successes, where x represents the binomial random variable and takes on the values 0, 1, 2, . . . , n. It is not too difficult for students to compute this probability using $C_{n,x}q^{n-x}p^x$ if they have had the necessary background. If they are able to do this, they can understand and will appreciate that as a random process is applied to a variable, it generates a sequence of values called the probability distribution of the variable. During the study of the binomial distribution, the students will understand that computations become very involved for large values of n. Hence, the normal

curve is introduced as an approximation to the binomial. An understanding of the normal curve can be developed by discussing frequency distributions of data resulting from measurements in the physical, social, and biological sciences. The students must realize that the binomial random variable is discrete and that the normal random variable is continuous.

The applications that are made of the normal distribution, as well as of the binomial distribution, have to do with questions of probability. The total area under the curve is equal to 1. It is too difficult for most high school students to find the $p(z \leq z_0)$ by using the equation of the normal curve. Nevertheless, they can appreciate that the formula is used in computing the values in the tables that they use. Also they are able to understand that they are finding the probability of x being in some range. Of course the change from the random variable x to the variable z is explained by discussing "standardizing" of the normal curve.

The students now have a background concerning probability and probability distributions. Such terms as population parameter, statistic, population, and random variable have been defined. The importance of a random sample has been emphasized. The problems of statistical estimation can be introduced at this time. Students can compute sample means and variances. They must realize that these are sample statistics which provide estimates of the true population parameters. From these sample statistics the student is able to calculate two numbers between which the parameter will lie with a known probability of error. The student thus becomes acquainted with a method of estimation called the method of confidence intervals.

To compute confidence intervals, the student must study the behavior of sample means as they vary under random sampling. These sample means generate a frequency distribution of the sample means called a sampling distribution. The various properties of the sampling distribution for sample means are then discussed. Likewise, the sampling distribution of sample proportions is discussed because of its applications in quality control. The sampling distribution of sample variances probably should not be covered in secondary mathematics.

After studying sampling distributions and statistical estimation, students have an excellent background for testing statistical hypotheses. Statistical reasoning enables the students to make decisions based on evidence obtained through proper sampling techniques. New definitions such as test criterion, test of a statistical hypothesis, critical region, significance level, and power of a test are developed at this time. The various possibilities for error are carefully considered. These possibilities arise from the fact that the hypothesis concerns the entire population, while the test criterion is derived from a sample from the population. After considerable discussion the various steps are developed, and ultimately the students come up with a definite procedure to be followed.

In this work it is extremely important that the students agree in *advance* when to accept a hypothesis and when to reject a hypothesis. This can be done very conveniently by agreeing to accept the hypothesis if the experiment yields a random event and reject the hypothesis if the experiment yields a rare event. A rare event is defined as one whose probability of occurrence is less than .05. A random event is one whose probability of occurrence is equal to or greater than .05. Because of the possibility of error, the students can appreciate why statistical inference is sometimes referred to as "probable" inference. This should be contrasted with the "exact" inferences of other types of reasoning.

Students now have some notion of the two aspects of statistical inference, estimation, and tests of hypotheses. Each of the different parts of statistical inference plays an important role in quality control. Hence, some work on quality control

should be included for high school students. It is not very difficult for students to recognize the presence of variation in seemingly identical manufactured pieces. Therefore, the construction of control charts by high school students very nicely summarizes the work on statistical inference.

Finally, the entire process of statistical reasoning should be reviewed and summarized for the students. This will be similar to the introduction but will now have more meaning for the students. Each student must understand that statistics cannot be studied without a knowledge of probability. The techniques of statistics are not isolated formulas, they enable the experimenter to draw conclusions about a population on the basis of the information given by a random sample. Furthermore, the student should realize that statistics is usually associated with an experiment which has been designed to investigate some particular problem situation.

Following a summary of statistical reasoning, many of the students will enjoy some elementary theoretical considerations. As far as statistical reasoning is concerned, the role of statistics in setting up an abstract model from observed results can be discussed. The role of statistics in deductive reasoning, the role played by the mathematical model in statistical reasoning, and statistical fallacies and faulty reasoning include other possibilities for discussion. A comparison of other types of reasoning should follow. Many of the students will also enjoy a discussion of how statistics enters into the procedures of the scientific method and even a discussion of science and philosophy.

In this article the writer has attempted to provide some ideas concerning the teaching of statistics in high school. The objectives and content included are merely suggested. The writer firmly believes that statistics for secondary-school mathematics must be directed at probability and its role in statistical inference. Many theoretical examples should be used involving coins, dice, and cards. Experimental results obtained by students should be compared with theoretical probability distributions for the particular random variable under investigation.

Much research is needed to determine how successfully students can learn the concepts of statistics and probability. Perhaps other teachers will experiment with these ideas. Let's begin to "modernize" the mathematics curriculum to include a study of statistics and probability!

"The true mathematician is always a good deal of an artist, an architect, yes, of a poet. Beyond the real world, though perceptibly connected with it, mathematicians have intellectually created an ideal world, which they attempt to develop into the most perfect of all worlds, and which is being explored in every direction. None has the faintest conception of this world, except he who knows it."—*A. Pringsheim.*

Finding two fractions having a special sum

P. H. NYGAARD, *North Central High School, Spokane, Washington.*

Your high school pupils may be interested in the results of this theorem, and you will be interested in the proof.

A LITTLE KNOWN and remarkable property of fractions is that any given fraction whose numerator is one less than its denominator can be obtained by adding two proper fractions, provided the given denominator is the product of two relatively prime integers. For instance, the fraction 39/40 can be obtained by adding two proper fractions, because 40 is equal to 5 times 8. Another illustration is 186/187, which equals 9/11+3/17.

The property above can be stated in the form of a theorem as follows: For any two given positive integers p and q, relatively prime to each other, there exist non-negative integers x and y, with x less than p and y less than q, so that

$$\frac{x}{p}+\frac{y}{q}=\frac{pq-1}{pq}.$$

A proof will be given for this theorem, utilizing the theory of congruences.

Assuming that the given equation is satisfied, the following are easily derived:

If $x=p-1$, $py=q-1$.

If $x=p-2$, $py=2q-1$.

If $x=p-3$, $py=3q-1$.

. . . . . .

If $x=1$, $py=q(p-1)-1$.

If $x=0$, $py=qp-1$.

It can be readily seen that there are a total of p values of x in this list.

Now set up the following p congruences, having each value of r as small as possible, but excluding negative values of r:

$$q-1\equiv r_1 \pmod p$$
$$2q-1\equiv r_2 \pmod p$$
$$3q-1\equiv r_3 \pmod p$$
$$\cdots\cdots\cdots$$
$$q(p-1)-1\equiv r_{p-1} \pmod p$$
$$qp-1\equiv r_p \pmod p$$

It will be shown that these values of r are all different from each other.

Assume that any two of them, say r_1 and r_p, are equal. If this were to be true, $q-1\equiv qp-1 \pmod p$, which is equivalent to $q\equiv qp \pmod p$, or $q(p-1)\equiv 0 \pmod p$. This can be true only if $q(p-1)$ is divisible by p. Since p and q are prime to each other by hypothesis, there is no factor in q which is a factor of p. Obviously $p-1$ is not divisible by p. Hence $q(p-1)$ cannot be divisible by p, so r_1 cannot equal r_p. A similar proof can be used for any two other values of r. Therefore all the above values of r must be different from each other.

The largest value r can have is $p-1$, because the modulus is p. The smallest value is, of course, 0. Since there are p values of r listed in the above congruences, and since none of them is a repetition of any of the others, it follows that the values of r are 0, 1, 2, 3, $\cdots$, $p-1$, not

necessarily in correct order. Therefore in the listed values of r there must be one, and only one, which equals 0.

Indicate the expression which happens to be congruent to 0 by $nq-1$. Then from the equations derived in the beginning of the proof, $py=nq-1$, and $y=(nq-1)/p$. Also $x=p-n$. These two formulas give the only values of x and y which satisfy the problem.

Suppose it is required to determine x and y so that $x/7+y/31=216/217$. Here $p=7$ and $q=31$. By actual computation the following congruences are found:

$$q-1\equiv 2 \pmod 7$$
$$2q-1\equiv 5 \pmod 7$$
$$3q-1\equiv 1 \pmod 7$$
$$4q-1\equiv 4 \pmod 7$$
$$5q-1\equiv 0 \pmod 7$$
$$6q-1\equiv 3 \pmod 7$$
$$7q-1\equiv 6 \pmod 7$$

Although it is not necessary to go beyond the congruence giving an r value of 0, the others have been found to show that r takes on every integral value from 0 to $p-1$, inclusive. Here $n=5$, so

$$y=\frac{nq-1}{p}=\frac{(5)(31)-1}{7}=22.$$

$$x=p-n=7-5=2.$$

It can be easily verified that $2/7+22/31=216/217$, and that other values are not possible.

Have you read?

DREIMAN, DAVID. "Fundamentals for Today's High Schools: A Layman's Point of View," *Educational Forum*, November 1957, pp. 95–103.

This author introduces his article with a statement that he does not want to booby-trap the schools nor does he wish to blast them for all the differences that have been brought about since he was in school. But he is frank in stating what he wants in a school. Among his requirements is developing the individual's power to think, to follow his judgments, to be courageous, inquisitive, adventurous, and public-spirited. He believes that good counseling will help and that principals should set the tone of the school so that it will stretch the mind. The individual should be so taught that he is willing to battle conformity, if this is best. The child's grounding in knowledge should be rich for all the years to come, and Mr. Dreiman believes that the teacher should be a person who will be remembered with gratitude. Instruction should raise the children so that they can be alone with their thoughts. "Teach eternity through your students" is this layman's plea.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

HEISEY, WILBUR. "Mathematics Capability and Employment Opportunities," *School Science and Mathematics*, December 1957, pp. 730–733.

This article is a summary of a study by the author on the relationship of the high school graduate's mathematics capabilities and his success in employment. The subjects were noncollege people. The method of the study was to interview employers. Illustrations of competence required are given. Some of the conclusions are: (1) Industry is using tests for selection; (2) Mathematics questions are prominent in the tests; (3) Mathematics in such tests is confined to four fundamental processes; (4) Degree of achievement in mathematics required varies from 33 per cent to 85 per cent; (5) A large percentage of students who fail do so because they lack mathematics skills.

There are several other conclusions and items of information you will want to study. I would suggest reporting this information to your noncollege-bound students for their consideration as they plan for their future careers.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The teaching of mathematical infinity

W. M. PEREL, *Texas Technological College, Lubbock, Texas.*

Infinity is a tricky word often misused in mathematics classes. It might be well to postpone its use until the students are old enough to comprehend how it is used in mathematics.

INFINITY is a word in the English language which is used by many people who are without training or interest in mathematics. The word does not have a very precise meaning to such people, but it does convey the idea of largeness of one kind or another. The photographer uses the word and the symbol associated with it to designate a distance greater than ten meters. The physical scientist sometimes associates it with a distance so large that phenomena at that distance may be ignored. The philosopher and the theologian both use the word in ways which, at times, seem to be ununderstandable. Madame George Sand wrote of Chopin, "He made an instrument speak the language of the infinite."

I do not propose to explain or criticize any of the above uses of the word "infinity." These uses are not mathematical and lie outside the scope of this paper. However, in mathematics a very precise meaning must be attached to each word used. No mathematician would attempt to define the word and concept out of context because infinity has several different meanings in mathematics. Each meaning will be clearly explained, but first it might be well to examine some of the many incorrect definitions which are given and the mathematical concepts from which they probably arise.

Probably the high school student and the freshman college student ought not to hear the word or see the symbol at all, at least in the mathematics classroom. As one of my former colleagues once put it, infinity is a word like the word sex. It really is not a bad word, but you have to be old enough before you can hear about it. To this I might add that in both cases what the student hears should depend upon his age.

Yet students hear the word "infinity" at a very early age and ask their teachers questions. Often they are told that infinity is a very big number or that it is the quotient obtained when an arbitrary number is divided by zero. Since any respectable algebra text specifically forbids division by zero, this statement can have no meaning to any thinking student. Yet teachers, and unhappily, some textbooks give it in the mistaken belief that some answer must be given and that the correct answer is too difficult.

Of course some writers would defend the statement, "a number divided by zero equals infinity," by using the statement, $\lim_{x\to 0}(a/x)=\infty$, but this statement not only has no meaning to a student who has not studied calculus, but is false. If a is an arbitrary positive number, then a/x assumes arbitrarily large positive or negative values according as x is assigned positive or negative values arbitrarily close to zero. A limit statement is just not able to handle this type of behavior.

$\lim_{x\to 0}(a/x^2)=\infty$ is a correct statement if a is positive, but even this statement does not justify defining ∞ outside of a limit statement.

In trigonometry classes, most students are taught that $\tan 90^\circ=\infty$. This false

equation appears in bold face type in all poor trigonometry texts—and also in many good ones. Sometimes the equation is followed by a paragraph explaining that it is false, but most students do not read the paragraph. No meaning at all can be correctly attached to tan90°. Perhaps we could use the symbol ∞ simply to mean "undefined," and one of my students once suggested this use, but we do not have the power to change the meaning of a mathematical symbol in use all over the world.

Second-year algebra students discuss infinite geometric series with no idea of what they are doing or why. Is it ever explained to the algebra student in either high school or college that adding up an endless array of terms can have a meaning? One even finds algebra students applying the binomial theorem to the case in which the exponent is a fraction or a negative number. The theorem has not been proved for exponents other than positive integers, nor can it be proved for such cases in an algebra class. Also, what is the student to make of the endless array of terms which the theorem gives him whenever the exponent is not a positive integer? No one explains convergence or divergence of infinite series to him. He adds up two, three, or four terms, according to his teacher's instructions, and assumes that his answer is approximately equal to what he started with. Frequently, problems are assigned in which the resulting infinite series diverges, so that the sum of the first few terms can have only an accidental relation to the correct answer. Students are already too prone to regard formal manipulation without understanding as the real mathematics, and every experienced teacher has a catalog of student errors which illustrate this misconception about the subject. Surely, we should do nothing to encourage this attitude among students.

The geometry student is sometimes told that infinity is the place at which parallel lines meet. Since parallel lines are defined to be lines which do not meet however far they are produced, it would appear that, according to this definition, infinity is a rather nebulous place where things which do not happen can happen. The idea of the intersection of parallel lines arises in projective geometry in order to permit the formulation of the principle of duality. This principle permits us to interchange the words "point" and "line" in the theorems about points and lines in plane geometry. Since two distinct points necessarily determine a line, it is desired that any two lines determine a point, that is, their point of intersection. It therefore becomes necessary to add points, usually called "ideal points," to the plane. If you wish, you may call them infinite points, but this use of the word "infinite" is not in any sense an explanation of any of the several other and more usual concepts associated with it.

While it is possible to make sense out of the idea of the intersection of parallel lines, no use can be made of this concept in the geometry usually taught to high school and first-year college students. Surely no one will claim that this concept will help the student to understand the ∞ he meets in calculus. Therefore, why mention it at all?

One use of the word "infinity" is in reference to the number of objects or elements in a set or collection of elements. Usually a finite set is defined to be one whose elements can be placed into one-to-one correspondence with the first n positive integers; that is, the elements of the set may be counted up until a definite integral value can be associated with the set. A set which is not finite is called infinite, or we say it has infinitely many elements. Thus the collection of all the people in the world is a finite set, and, whatever poets may say, the collection of grains of sand on a beach is also finite. A set may have very many elements and still be a finite set. However, there are

infinitely many positive integers, which is extremely important. It enables us to count the elements of any finite set, and incidentally means that our number system will always be able to represent the national debt, no matter how large it becomes. It has been shown that there are infinitely many prime integers, infinitely many rational numbers, and infinitely many real numbers. The fact that some infinite sets may be larger than others need not concern us here.

Another common use of the word and symbol is in limit statements. In calculus, the student meets the equations,

$$\lim_{x\to\infty} f(x)=h, \qquad \lim_{x\to-\infty} g(x)=k,$$

$$\lim_{x\to a} F(x)=\infty, \text{ and } \lim_{x\to b} G(x)=-\infty.$$

These equations state respectively that $f(x)$ takes on values arbitrarily close to the number h if sufficiently large positive values are assigned to x; that $g(x)$ takes on values arbitrarily close to the number k if sufficiently large negative values are assigned to x; that $F(x)$ takes on arbitrarily large positive values if values sufficiently close to the number a are assigned to x, and that $G(x)$ takes on arbitrarily large negative values if values sufficiently close to the number b are assigned to x. More precise definitions may be found in good calculus texts. It should be stressed that we do not replace the first equation by $f(\infty)=h$, because we are only allowed to substitute numbers into functions and ∞ is not a number. Also, we never write $F(a)=\infty$, because function values are always numbers when they exist at all.

Some mathematicians refuse to dignify the symbol ∞ by writing the third and fourth equation at all. However, I feel that it is useful to distinguish between the behavior of $1/x^2$ and of $\sin(1/x)$ for values of x near zero.

The more advanced student will see $\int_a^\infty f(x)dx$. This collection of symbols has no meaning until it is given one. Assuming that the definition of the definite integral is known, $\int_a^\infty f(x)dx$ is defined to be $\lim_{y\to\infty} F(y)$, where $F(y)=\int_a^y f(x)dx$. Thus this concept is reduced to the previously mentioned one.

Limit statements are also made with reference to sequences. A sequence is simply a correspondence between numbers and the positive integers. Usually we write a sequence symbolically, a_1, a_2, a_3, $\cdots$. We say that a number a is the limit of this sequence, written $\lim_{n\to\infty} a_n=a$, if the numerical difference between a and a_n can be made arbitrarily close to zero if n is large enough, that is, if a_n is sufficiently far out in the sequence. Actually this idea of limit is no different than that previously discussed case, $\lim_{x\to\infty} f(x)=h$, except that in this case x is restricted to the positive integers. Here, as in the preceding discussion, a more precise definition of limit may be found in the literature. It should be pointed out that for many sequences there is no number a with the property described above. In such cases, the sequence does not possess a limit.

We next consider infinite series. We write

$$c_1+c_2+c_3+\cdots=\sum_{j=1}^{\infty} c_j.$$

This collection of symbols does not always have a meaning. In all cases, however, we speak of the sequence of partial sums, defined by $s_1=c_1$, $s_2=c_1+c_2$, and in general $s_n=c_1+c_2+\cdots+c_n$. When $\lim_{n\to\infty} s_n=s$ exists, the infinite series is said to be convergent and to have the "sum" s. If the sequence of partial sums does not have a limit, then the infinite series is called divergent, and we do not usually attach any meaning to it at all. Certainly we never speak of its "sum."

If the binomial theorem is applied in the case in which the exponent is not a positive integer, an infinite series may be produced formally. Also one can obtain an infinite series by means of the Taylor expansion of functions. Euler and other

eighteenth-century mathematicians used such series formally without regard to questions of convergence or divergence, but it has now been more than a century since Gauss correctly interpreted their work, and was able to show in which cases it was correct.

No discussion of infinite series should be attempted in the classroom unless the teacher is willing to go through some such explanation as has been given above. Without it, the student will not even be able to tell whether what he is doing has meaning. Personally, I would prefer to leave such discussions entirely out of both high school and college algebra.

The word "infinity" is used at the real and complex variable level in graduate mathematics courses in other ways, which are outside the scope of this paper. These ways are not likely to confuse the undergraduate or high school student.

In conclusion, let me urge all mathematics teachers to avoid mentioning infinity as long as possible and never to define it incorrectly. Frequently the words "endless" or "unending" will serve as substitutes and I strongly advocate their use. Let us make sure our students are "old enough" before we introduce them to adult mathematics.

The Euler identity

ROBERT C. YATES, *College of William and Mary, Williamsburg, Virginia.*

A concise way to establish a well-known identity.

ONE OF THE MOST STARTLING (and useful) mathematical relations is the identity[1]

$$e^{ix} \equiv \cos x + i \cdot \sin x. \qquad (1)$$

To establish the identity, let us first agree that the symbol e^{ix} represents a function of x whose value is *one* for $x=0$ and whose derivative is $i \cdot e^{ix}$. Then consider the determinant

$$\begin{vmatrix} e^{ix} & \cos x & \sin x \\ ie^{ix} & -\sin x & \cos x \\ -e^{ix} & -\cos x & -\sin x \end{vmatrix}. \qquad (2)$$

Since the last row is the negative of the first, this determinant has value zero for all x. Expanding by elements of the first row:

$$e^{ix} - \lambda \cos x - \mu \sin x \equiv 0 \qquad (3)$$

where λ and μ are:

$$\begin{aligned} \lambda &= -i \cdot e^{ix} \sin x + e^{ix} \cos x, \\ \mu &= \quad i \cdot e^{ix} \cos x + e^{ix} \sin x. \end{aligned} \qquad (4)$$

If we differentiate λ and μ, we find

$$\begin{aligned} \lambda' &= -i \cdot e^{ix} \cos x + e^{ix} \sin x + i \cdot e^{ix} \cos x \\ &\quad - e^{ix} \sin x \equiv 0, \\ \mu' &= -i \cdot e^{ix} \sin x - e^{ix} \cos x + i \cdot e^{ix} \sin x \\ &\quad + e^{ix} \cos x \equiv 0. \end{aligned}$$

Thus λ and μ are constants. To evaluate, we set $x=0$ in equations (4). We find

$$\lambda = 1, \qquad \mu = i,$$

and the identity is accordingly established.

[1] Although the Euler identity is customarily established in the Series section of a Calculus course, the presentation here makes it acceptable at an earlier date. Another derivation may be found displayed in two books: *Differential Equations* by Walter Leighton and *Differential Equations* by Robert C. Yates, both published (without collusion) by McGraw-Hill Book Co., Inc., New York, in May, 1952. In each book the derivation appears as a footnote on page 50! It may also be found in a short paper, "A Brief Derivation of the Euler Exponential Form," by Robert C. Yates, *Mathematics Magazine*, XXIV (1950), 57.

These eighth graders don't hate arithmetic

GEORGE R. ANDERSON, *State Teachers College, Millersville, Pennsylvania.*

This report gives the results of a questionnaire administered to a group of eighth-grade pupils in Pennsylvania. What would your eighth graders say about mathematics?

THE NOTION that mathematics is the most disliked school subject has become widely accepted. This dislike, it would seem, is shared by both teachers and pupils. The lack of mathematical ability is a frequent subject for the cartoonist. Indeed, some people point with questionable pride to their inability to make correct entries in a checkbook.

In view of the number of statements supporting the above assertions, it was with some misgiving that the writer attempted to determine the attitude of 534 eighth-grade pupils toward their major subjects. This was part of a larger study designed to measure the efficacy of a kit of sixteen visual-tactual devices in a unit of arithmetic. A copy of the questionnaire appears on page 268.

This questionnaire was administered to all eighth-grade pupils in three suburban junior high schools near Lancaster, Pennsylvania. The students were taught by five teachers. To avoid any possible teacher-pupil bias, the questionnaire was given by the guidance personnel in each school. It was first given in September 1956, during the second week of school. It was repeated in February 1957, near the beginning of the second semester. About half the students (264) comprised an experimental group which used a kit of sixteen visual-tactual devices in the study of arithmetic. The remainder of the group (270) made up the control group. This latter group did not use the devices. Both groups used the same textbook and used the same tests throughout the semester. While primary interest was in the rank order assigned the various subjects, it was believed that by asking students to select the best liked and the least liked subjects a more valid rating might be expected for the remaining two subjects.

To give weight to first, second, third, and fourth place rankings, an index number was devised. Calculation of the index number for mathematics as of September 1956 will illustrate the method employed. The 264 pupils indicated a preference for mathematics as follows:

RANK	NUMBER OF PUPILS	WEIGHTING FACTOR	PUPILS × WEIGHT
First	115	4	460
Second	64	3	192
Third	47	2	94
Fourth	38	1	38
	264		784

Name.....

Section.....

School.....

The four major subjects you are studying this semester are: English, Geography, History, and Mathematics.

Please answer the following questions carefully.

1. Which one of the four major subjects do you like *best?* (best) .....
2. Which of the four do you like *least?* (least) .....
3. Rank all four subjects in order of your choice. Place the one you like *best* in number 1 position. Place the one you like *least* in number 4 position.

(best) 1. .....

2. .....

3. .....

(least) 4. .....

4. What made you select the subject you stated you liked best? Write your reasons below.

Dividing the *sum* of the numbers in the last column by the number of pupils in the group yields an index number of 2.97. This may be regarded as an average ranking of the subject. If all pupils placed a subject in first place, the index number would be four. If all ranked it in fourth place, the index number would be one. Therefore, these index numbers may be conceived to represent points along a preference scale on which four is the highest attainable and one is the lowest attainable. All the index numbers are quite close together. They range from 1.99 to 2.97. If the subjects were ranked at random, the index number might be expected to be 2.50. All subjects fell within plus or minus one-half rank of this chance rating. A summary of all rankings appears in the table on page 269.

The relatively high rank assigned to mathematics by all groups in the study seems to cast some doubt on the commonly held notion that pupils do not like that subject. It should be observed that all groups placed it either first or second. It did suffer a slight decline from September to February, but this is probably explained by the rigorous time limits imposed by the study and by the more than usual number of tests given. It is interesting to note that the use of the visual-

TABLE 1

PUPIL PREFERENCE FOR FOUR MAJOR SUBJECTS

SUBJECT	EXPERIMENTAL $N=264$		CONTROL $N=270$		TOTAL $N=534$	
	INDEX NO.	RANK	INDEX NO.	RANK	INDEX NO.	RANK
			SEPTEMBER 1956			
Mathematics	2.97	First	2.68	Second	2.82	First
English	2.57	Second	2.32	Third	2.44	Second
Geography	2.00	Fourth	2.75	First	2.38	Third
History	2.47	Third	2.23	Fourth	2.25	Fourth
			FEBRUARY 1957			
Mathematics	2.81	First	2.47	Second	2.64	First
English	2.70	Second	2.51	Third	2.60	Second
Geography	1.99	Fourth	2.83	First	2.42	Third
History	2.53	Third	2.18	Fourth	2.35	Fourth

tactual devices had no apparent effect on sustaining interest in the subject. In the experimental group, mathematics suffered about the same decline as in the control group.

Is this an unusual group of pupils? The writer feels that they are representative eighth-grade pupils. Could it be that the publicity accorded mathematics and science in recent years is having a favorable effect on the attitude of pupils toward mathematics? If this is correct, perhaps we can look forward to more nearly meeting the need for widespread knowledge of mathematics in our increasingly technical society.

What's new?

BOOKS

SECONDARY

Plane Trigonometry, Frank A. Rickey and J. P. Cole. New York, Dryden Press, Inc., 1958. Cloth, x+260 pp., $2.90.

COLLEGE

Colloquium Publication, Vol. 31 (rev. ed.), "Functional Analysis and Semi-Groups," Einar Hille and Ralph S. Phillips. Providence, R. I., American Mathematical Society, 1957. Cloth, xii+808 pp., $13.80.

Elements of Modern Abstract Algebra, Kenneth S. Miller. New York, Harper and Brothers, 1958. Cloth, vii+188 pp., $5.00.

Fundamentals of Mathematics (rev. ed.), M. Richardson. New York, The Macmillan Company, 1958. Cloth, xviii+507 pp., $6.50.

MISCELLANEOUS

International Series of Monographs in Pure and Applied Mathematics, Vol. 2, "Circles," D. Pedoe. New York, Pergamon Press, Inc., 1957. Cloth, x+78 pp., $3.75.

The Tree of Mathematics, Glenn James (ed.). Pacoima, California, The Digest Press, 1957. Cloth, xvii+403 pp., $6.00.

Time to stand together

WILLIAM G. CARR, *Executive Secretary,*
National Education Association, Washington, D.C.

The Executive Secretary of the NEA tells why your NEA membership dues were increased. Let's stand together.

NO ELABORATE FORMULA or lengthy computation is required to explain the 1957 increase in NEA dues. Only two major factors are involved: (*a*) mounting costs, or inflation; and (*b*) a larger and better service program.

More services @ higher prices = increased dues.—You may have noticed that a loaf of bread or a pair of shoes now cost more than you once paid! So it is with association costs. Since 1948, for example, the price of mimeograph paper has jumped 23%. The secretary who in 1948 was employed for $2400 must now be paid at least $3500. Each new typewriter costs 60% more than the old one it replaces. By 1957, the NEA found it impossible, on the $5 dues established in 1948, to keep staff salaries in line with those of competing agencies and maintain the services which it had been giving. But all the time *demands for more services* were pouring in.

So, after two years of discussion and planning, 6000 delegates at Philadelphia by a vote of more than 4 to 1 voted to increase NEA dues to $10. At the convention, too, the broad outline of an expanded service program was developed and approved.

Ways the NEA is extending and improving its services.—Immediately after the Philadelphia convention in July 1957, new services began to be introduced and, through the intervening months, have been added as rapidly as possible. Listed here are examples of some additions and improvements in service accomplished prior to December 1, 1957.

A salary consultant has been employed to work full-time with state and local association leaders and salary committees.

More *field work* is being done. Three new regional representatives are working out of San Francisco, Cheyenne, and Atlanta, respectively. The number of appointments accepted by the Washington staff has increased about 25 per cent.

Press contacts have been increased, resulting not only in excellent newspaper coverage of school news, but also in some unusual coverage in national magazines. For specific examples, see *Reader's Digest*, July 1957, inside back cover; *Saturday Evening Post*, September 21, the "Yes" article by Herbert Brown; and *Look*, November 26, page 40.

An experienced TV representative has been employed, with offices in New York City, to work with radio and TV networks in collaboration with the NEA's Washington office of *Press & Radio Relations*. Between September and December 1957, the NEA had co-operated in some way with the producers or networks on 62 radio and TV shows. One of these network programs alone estimates that in three days its impact was "250 million listener impressions."

More research work is made possible by new IBM equipment: more studies; more

extensive analysis of data; more prompt reporting. The *Research Bulletin* is being improved in format to make it more practical.

Legislation is being sought *to authorize income-tax deduction of teachers' summer school and other necessary professional expenses.* Administrative rulings have denied this right to teachers so far. For this reason, the NEA will vigorously push for clear definition in the law which will give teachers the same privilege now granted to many other professions.

The NEA Journal *is being improved* in both format and content. There will be two large 80-page issues this year. The *NEA News* is being published more often and distributed more widely. *A new director of publications has been employed* to coordinate production and promote wider use of the NEA's 1400 publications currently available.

Improvement of instruction is getting additional emphasis. A special study is under way on the education of the gifted. New consultants have been employed for kindergarten-primary education, elementary science, educational TV, and rural education. Twelve regional conferences on various instructional problems were on the 1957 autumn schedules of the NEA and its departments.

Committee work can be more effective when funds are available for meetings, printing costs, and other necessary expenses. The NEA has 30 committees and commissions, including important joint committees with such groups as the Magazine Publishers, National Congress of Parents and Teachers, and National School Boards Association.

The foregoing illustrations indicate only some of the major *additions to* and *improvements in* the NEA program. There has been only casual and incidental reference to a few phases of the long-established, on-going program of the Association. Other extensions and improvements in service are planned and will be inaugurated as rapidly as circumstances permit. How much can be done, and how soon, depends on the number of educators who enroll in the Association and participate in its work.

Among the steps we hope to take in the near future are: employment of a UN observer for education; addition of a specialist on higher education; and employment of a staff member to work full-time with institutions engaged in teacher education.

Relation of NEA to its departments.—In legislation, in public relations, in its research and information services, the NEA serves all teachers, at all grade levels, and in all subject areas. Members of NEA departments, however, have a further stake in the success and strength of the parent organization. There is great mutual advantage when a department has its headquarters at the NEA Education Center in Washington. Its presence there gives a department status and prestige. But far more important, any department with its headquarters at NEA has at its command a vast reservoir of information and expert consultant service on almost every phase of education. Through office space, services, and sometimes by direct financial assistance, the NEA helps the departments do their jobs. Last year, NEA contributed money and services estimated at more than $250,000 to the various departments.

A strong NEA+strong departments= good education and gains for our profession.—As teachers, we must all stand together. The opportunities we want for boys and girls and the improvements we want for our profession must be attained by teamwork. There is no other way.

On teaching dihedral angle and steradian

HOWARD FEHR, *Teachers College, Columbia University, New York, New York.*

An extension of the method of defining a plane angle analytically to the definitions of a dihedral angle and the steradian.

A LINE IN A PLANE divides the entire plane into two regions called half-planes. In three-dimensional space, let two half-planes not in the same plane have the same initial line or edge. Figure 1 is an illustration of this situation where α and β are half-planes and l their common edge. The figure consisting of α, β, (and l) is called a dihedral angle. It divides the entire three-dimensional space into two distinct regions with the following properties:

1. One region is such that any two points in it can be joined by a straight line segment which does not intersect the dihedral angle. (See P and Q as an illustration.) This region is called *convex*.
2. The other region has points in it which can be joined by a straight line segment which will intersect the dihedral angle. (See Z and W as an illustration.) This region is called *reflex*.

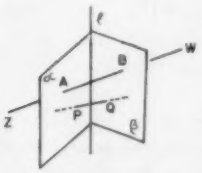


Figure 1

Thus all the points of space have been separated into three sets, namely:

1. the dihedral angle or the set of points on two half-planes and their common edge
2. the set of points of the convex region bounded by the dihedral angle
3. the set of points of the reflex region bounded by the dihedral angle.

If the convex region is considered the interior of the angle, we say the angle is convex. If we consider the reflex region as the interior of the angle, we have a reflex dihedral angle. If the two half-planes lie in the same plane, both regions are convex, and for either region the angle is a straight dihedral angle.

ANALYTIC DEFINITION OF DIHEDRAL ANGLE

Consider a set of rectangular co-ordinates in three dimensions, the same scale on each axis, so that the system

$$\begin{cases} z = ax + by + c \\ z = dx + ey + f \end{cases} \qquad a, b, e, d, \neq 0$$

defines two planes. If $ae \neq bd$, the planes intersect in a line. We shall use this system to discuss the dihedral angle.

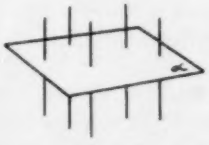


Figure 2

A single plane divides all space into two regions, both convex, each called a half-space (Fig. 2). Thus a half-space is to a

plane as a half-plane is to a line. We call the plane a *face* of the half-space. The two planes of the above system of equations form four half-planes (α, β, γ, δ) (Fig. 3). These half-planes, considered two at a time, form six dihedral angles, four of which bound convex regions less than a half-space. Figure 3 shows the four convex dihedral angles, reading counterclockwise, (α, δ), (δ, γ), (γ, β), and (β, α).

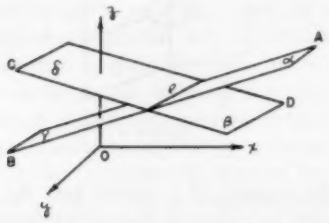


Figure 3

We can now designate the interiors of these convex dihedral angles as the intersection of two properly selected half-spaces. Let the equation of plane AB be designated by $z=ax+by+c$ and that of the plane CD by $z=dx+ey+f$. Figures 4 to 7 show the four possible intersections of either half-space of plane AB with either half-space of plane CD. The interior of a convex dihedral angle is thus defined as the intersection of two selected half-spaces. The dihedral angle is the set of points on the half-planes bounding the interior. The remaining points of the plane constitute the exterior of the dihedral angle.

(γ, β)

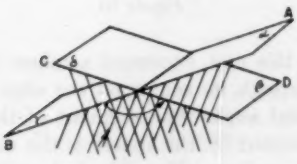


Figure 4

$$\{(x, y, z) \mid z<ax+by+c\} \cap \{(x, y, z) \mid z<dx+ey+f\};$$

(β, α)

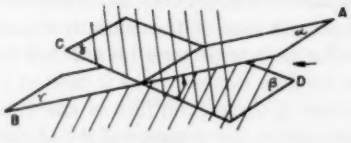


Figure 5

$$\{(x, y, z) \mid z<ax+by+c\} \cap \{(x, y, z) \mid z>dx+ey+f\}$$

(δ, γ)

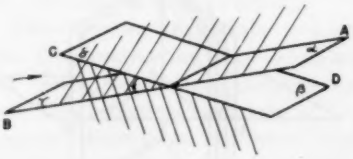


Figure 6

$$\{(x, y, z) \mid z>ax+by+c\} \cap \{(x, y, z) \mid z<dx+ey+f\};$$

(α, β)

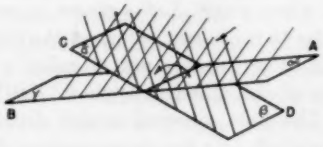


Figure 7

$$\{(x, y, z) \mid z>ax+by+c\} \cap \{(x, y, z) \mid z>dx+ey+f\}$$

MEASUREMENT OF THE DIHEDRAL ANGLE

To measure a dihedral angle, we select a dihedral angle of arbitrary size as a unit. This unit dihedral angle is constructed in the interior of the angle to be measured with the edge and one face of the unit coincident with an edge and face of the angle. This process is repeated as shown in Figure 8. The faces are then numbered in cardinal order, naming one face of the given dihedral angle zero and continuing through the interior of the angle until the second face of the dihedral angle is

reached. (If the second face of the last unit dihedral angle does not coincide with the second face of the measured angle, a smaller unit angle may be selected to obtain as close a measure as is desired.) The number of the second face of the dihedral angle is then the measure of the angle. If a unit angle is selected so that the measure of a straight angle is 180°, we call this unit a degree of dihedral angle.

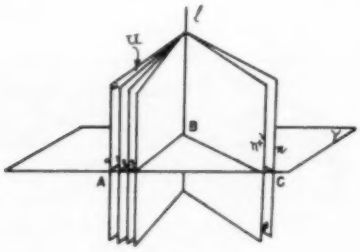


Figure 8

Another convenient way to measure a dihedral angle is to relate it to the measure of a plane angle. Let a plane γ, perpendicular to the edge l, intersect the dihedral angle. The intersection is called a plane angle of the dihedral angle ($\angle ABC$, Fig. 8). The unit dihedral angles divide the plane angle into the same number of *equal* plane angles as there are dihedral units in the dihedral angle. Hence the measure of this plane angle is the same number as the measure of the dihedral angle. Since all the plane angles of a given dihedral angle, formed in the manner described here, are equal, we can say: A dihedral angle has the same measure as its plane angle.

Just as we consider rotation of a half-line in a plane to form a plane angle through its initial and terminal positions, we can also conceive of a rotation of a half-plane in space about its edge to form a dihedral angle through its initial and terminal positions. If we think of the half-plane as making a complete rotation (i.e., to rotate so as to terminate at the starting position), we have the feeling that the infinite half-plane has traversed the whole of three-dimensional space. If the dihedral degree is the fundamental unit of measure, we can look upon 360° as the measure of all space, just as 360° of a plane angle is the measure of the entire plane. (Fig. 9.)

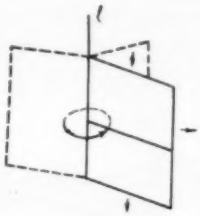


Figure 9

The surface of a sphere has the same relation to space as the circumference of the circle has to a plane. In traversing the entire surface of a sphere, one has the feeling that all possible extensions into space are covered. That is, the surface of a sphere can serve as a finite measure for the angular encompassing of all three-dimensional space. Thus we can use the surface of a sphere to measure a dihedral angle.

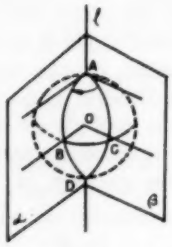


Figure 10

To this end, construct a sphere of unit radius with its center on the edge of the dihedral angle. The segment of the edge intercepted by the sphere is the diameter of the sphere. The dihedral angle thus intercepts a segment of the surface of the sphere which is a lune ($ABDCA$ in Fig. 10). The angle of the lune is equal, in degrees, to the plane angle of the dihedral

angle. Hence the dihedral angle is measured by the lunar angle on the associated sphere.

Steradian

Since a complete (360°) dihedral angle would encompass the whole sphere, and any angle less than 360° intercepts a lunar area in the same ratio to the area of the sphere as the angle of the lune is to 360°, we shall use the area of the lune as a measure of the dihedral angle (or as a measure of the portion of space interior to the dihedral angle). The area of a unit sphere is 4π square units. If the angle of the lune is A degrees, the area of the lune in square units is $(A/360)\times 4\pi$ or $A\pi/90$. If A is the number of degrees in the dihedral angle, we can now say its measure is $\pi A/90$.

One may well ask what name is given to this new unit of dihedral angle. Of course it is a number of square units of spherical surface. Recall that when the circumference of a unit circle was used to measure a central angle, the angle intercepting an *arc* of unit length was selected as the fundamental unit and called a radian. By analogy, we select a dihedral angle intercepting an *area* of one square unit on the associated sphere as the fundamental unit and call it a *steradian* (solid radian). Then a complete angle has 4π steradians. The measure of a dihedral angle of $A°$ is $(A\cdot\pi)/90$ steradians. If, however, we measure the dihedral angle in radians, say θ, then the area of the intercepted lune is $(\theta/2\pi)\cdot 4\pi$ or 2θ square units. In this case, the measure of the dihedral angle is 2θ steradians.

Summary

Two half-planes with the same initial edge form a dihedral angle. If the half-planes are not coplanar they divide space into two regions, one convex, the other reflex. If the convex region is the interior of the angle, the angle is said to be convex. If the reflex region is the interior of the angle, the dihedral angle is reflex.

A plane divides space into two half-spaces. Two intersecting planes have associated with them four distinct half-spaces. The intersection of two of these half-spaces determines a convex region which is the interior of a dihedral angle.

A dihedral angle, or more properly the interior of a dihedral angle, is measured by the number of unit dihedral angles it contains, or by the measure of its plane angle. A dihedral angle is also measured by the angle of the lune it intercepts on a unit sphere with its center on the edge of the dihedral angle. It is also measured by the area of the intercepted lune. A steradian is the dihedral angle which intercepts on its associated sphere an area of one square unit. If the angle of the dihedral angle is θ radians, then its measure in steradians is 2θ. The dihedral degree and the steradian are also measures of space coverage.

"I should rejoice to see . . . morphology introduced into the elements of algebra."—*Sylvester.*

"Applications" in freshman courses: an example[1]

ROBERT L. DAVIS, *University of Virginia, Charlottesville, Virginia.*

An application of some modern mathematical concepts to a simple situation in decision-making.

Introduction

One way some modern courses are moving is toward more examples, toward introducing mathematical theory as far as possible by means of real problems that the theory can handle. To be sure, this is only one way, and there are a great many doing very nearly the opposite; but the approach through applications is what our Committee on the Undergraduate Program in the Mathematical Association of America thinks is typical of its best work.

There is a danger, in trying to tell people about this approach through applications, of sounding like the publisher's blurb for a second-rate textbook. We have all heard a great deal about "concrete illustrations," "the operational approach," "real-life applications," and so on. Perhaps a better way to get the right idea across is to give an example.

My choice of example is conditioned by two things. I want to pick an example which is simple and quickly understood, and yet which illustrates at least in rudimentary form several mathematical ideas. And I would like the mathematics to be of the kind that our committee keeps insisting is of first importance: the simple mathematics of sets. We believe this kind of mathematics is not only important in modern mathematics and modern ways of looking at classical mathematics, but also that it is fundamental in applications of mathematics to all kinds of sciences. Hence it is a virtue, in a simple example like the one I will give, that the example makes use (even if in a very elementary way) of the ideas of set and subset, of inclusion and chains of subsets under inclusion, of a very simple set function defined on all the subsets of a given "universal" set, and of a point function which summarizes in handier form the information of the set function.

The example is that of the *power index* for a decision-making body. It is derived from the "Shapley value" of game theory. Shapley himself, with the economist Martin Shubik, was the first to point out this simple application of his "value" function. The version given here is derived from one which was prepared by A. W. Tucker of our committee.

When students come to this application in our course, they have just covered the basic properties of, and operations with, sets. Thus they have learned in the first chapter about Venn diagrams, the symbols and concepts of set algebra, and the Boolean properties. The second chapter treats the natural numbers and mathematical induction, chiefly as preparation for the subject of Chapter III: counting in sets. This last includes much of the combinatorial analysis that we are used to seeing in probability books, with certain revisions in the standard cumbersome and ambiguous language. (The committee very much likes this material for its im-

[1] Extracts from a talk given to the National Council of Teachers of Mathematics at Philadelphia on March 30, 1957, under the title "Freshman College Mathematics Today and Tomorrow."

mediate uses as well as for the drill and review it provides in manipulatory skills.) All of this has been presented from examples: almost every section begins with the thorough description of some such example, and only after analysis of a particular way of solving the problem are any general definitions or propositions brought in.

Decision-making bodies

The *power index* is a function defined on a set called a *decision-making body*. Committees, boards of directors, stockholder groups, and electorates are all decision-making bodies. What distinguishes them is that they have some means of coming to decisions; a simple way of describing this in terms of sets is borrowed from game theory. Before we give this description, let us consider some examples.

1. $U = \{1, \cdots, n\}$ is a set of n people in which every decision is a dichotomy decided by simple majority vote. In this case the "rules of the game" suggest that all the individuals have equal power to affect the decision of the group.
2. Again $U = \{1, \cdots, n\}$ but here 1 has two votes, while each other member of the group has only one.
3. U is the set of three stockholders of a company holding 49 per cent, 49 per cent, and 2 per cent of the stock respectively. Decisions require the vote of a majority of all stock.
4. U is the Security Council of the United Nations. (We are here concerned only with the abstract situation described by the laws of the Charter and not with the political realities. To make clear what an abstract analysis can and cannot do in such real problems as this will be an important part of teaching a freshman mathematics course.)

In all of these cases and in many more there is a clear usefulness in being able to assess in advance the power situations described by the rules. The power index is a way of doing so, and in fact it is essentially the only one. That is, the function we will define is the only nonnegative function whose values add up to 1 which satisfies a certain natural symmetry condition and a somewhat more complicated "additivity" condition. Any other function which satisfies these two criteria must differ from ours only in origin and a scale factor.

Notice that there is clearly no simple guide to guessing the relative power of the members in case (4), while first guesses are likely to be wrong in (2) and possibly in (3). In (2) it would be careless to say that 1 is twice as powerful as any other member just because he has twice as many votes. For if $n=3$, then 1 with his two votes can block any decision desired by the other two. Case (3), perhaps, would fool no one; it is clear that in decision-making the three stockholders are equally powerful. (Again, this kind of example gives an excellent chance to make clear the mode of abstraction: no one says the 2 per cent stockholder feels as important or makes as much money as the others; this abstraction is solely concerned with power in influencing decisions.)

The power index

To introduce our definition we must define the decision-making body more precisely. The game-theory method is very simple. We call all subsets of U "coalitions." This suggests that we are thinking of the players of a given subset as cooperating to support a given decision. We will know the rules of the decision process precisely when we can say, for each subset, whether or not it is a strong enough coalition to decide the issue in its favor.

Now it would not do just to pick out in this way any set of subsets as the "winning coalitions." We would want to have the set of winning coalitions satisfy the requirements (a) that adding more members to a winning coalition gave a new winning coalition, and (b) that the complement of a winning coalition was not a winning coalition. It would also be useful to insist that (c) there be at least one winning coalition.

To give the students a foretaste, in the simplest imaginable situation, of nobler things to come we may phrase this in terms of a set function F taking just the two values W and N, say, or 1 and 0. (It is a nice question whether to illustrate with the former pair that function values need not be numbers, or with the latter that numbers may be used without any cardinal or quantitative significance.)

Definition 1: A (finite) set U is a *simple decision-making body* if there is a set function F defined on subsets of U such that for any sets S and T:

1. If $F(S)=1$, then $F(S \cup T)=1$, for any T;
2. If $F(S)=1$, then $F(U-S)=0$;
3. There is a T with $F(T)=1$.

Now Shapley's method of assigning a power index for a given member is to look, conceptually, at all possible ways of building coalitions by adding one member at a time, starting with the empty set and winding up with the universal set. Then the power index assigned to the member will depend on the chance that that member was "pivotal" in making a coalition a winning one: that his joining was critical to the success of that coalition.

Notice that in most cases, where there will be no coalitions explicitly forbidden by the rules of the game, all possible ways of building up coalitions will correspond in a natural way to all possible ways of ordering the n members of the set U. That is, given any such ordering $(i_1, i_2, \cdots, i_n)$, we can define a chain of coalitions $S_0=\phi$, $S_1=\{i_1\}$, $S_2=\{i_1, i_2\}$, $\cdots$, $S_n=\{i_1, \cdots, i_n\}$. Our three conditions guarantee that somewhere in this chain there is a first winning coalition: call it $S_p=\{i_1, \cdots, i_p\}$. Then all preceding coalitions are nonwinning, and all later ones are winning. Thus we can think of the chain of coalitions as it appears at the foot of this page.

Under these circumstances member i_p is "pivotal" in determining that the set S_p is a winning coalition. In this way we call any member i_p a *pivot* in the ordered sequence $(i_1, \cdots, i_{p-1}, i_p, i_{p+1}, \cdots, i_n)$ if the set $\{i_1, \cdots, i_{p-1}\}$ is not a winning coalition, but the addition to it of i_p makes it into a winning coalition.

Definition 2: The *power index* of a member of a decision-making body is the number of ordered sequences in which he is pivot, divided by the total number of ordered sequences of the members.

Applications

The device of considering all ordered sequences was used solely for conceptual aid. In fact, once we have the power index established, we can forget about most of these sequences; in view of the rapid increase of the number of ordered sequences, this is a prime desideratum.

Suppose, for instance, that $S_{p-1}=\{i_1, \cdots, i_{p-1}\}$ is a nonwinning coalition which is turned into a winner with the addition of member i_p. Then any ordering of the members in which the first $p-1$ members are permuted among themselves and the last $n-p$ among themselves will again give rise to i_p's being a pivot. Now our students are prepared by the previous chapter to tell us that there will be $(p-1)!(n-p)!$ such orderings among the total of $n!$ orderings of all the members. Thus they are able to make a considerable saving in computation. In this way they can show that the power index may be computed as follows:

Call a coalition S_p *minimal with respect to member* i if (1) S_p is a winning coalition, (2) i belongs to S_p, and (3) the coalition

$$\underbrace{S_0 \subset S_1 \subset \cdots \subset S_{p-1}}_{\text{nonwinning}} \subsetneq \underbrace{S_{p+1} \subset \cdots \subset S_{n-1} \subset S_n}_{\text{winning}}$$

(i_p added)

$S_p - \{i\}$ is not winning. Then the power index, let us call it π, can be computed as

$$\pi(i) = \sum_{S_p} \frac{(p-1)!(n-p)!}{n!},$$

the sum taken over all coalitions S_p which are minimal with respect to i; p is the cardinality of S_p.

Even with this short cut it can well be hard to compute the power index. One further simplification is the fact that the function π has a certain symmetry. If two members' positions are indistinguishable (in the rules of the game), then they must have the same power index.

From this last rule alone we can tell that in our first example (above) all members must have the same power, since it is clear that all the power index values must add up to 1. We can conclude in this case that $\pi(i) = 1/n$ for each member i. In example (2) the symmetry rule tells us that all the members other than 1 have the same value. (In fact, for $n = 3$, it is easy to compute $\pi(1) = 2/3$ and $\pi(2) = \pi(3) = 1/6$.) The symmetry rule only tells us, in the third example, that the first two members have the same power index; it happens in this case that their index is the same as that for the third member.

The hardest of these examples is that of the Security Council. Each of the short cuts mentioned above can be used to work this one out in class. First, observe that for a measure to pass the Council, it must be supported by seven members, including all of the "Big Five" nations. Therefore, for any one of the six smaller nations there are just five winning coalitions that are minimal with respect to that nation: each of these winning coalitions consists of the "Big Five," the given small nation, and one of the five remaining small nations. Hence by our formula for computing the power index in terms of coalitions minimal with respect to a given member, we have for any small nation x:

$$\pi(x) = \sum \frac{(p-1)!(n-p)!}{n!} = 5\left[\frac{6!4!}{11!}\right]$$

$$= \frac{1}{22(21)} = 1/462.$$

Then, using symmetry again and the fact that the index values add up to 1, we can also compute the power index of any of the "Big Five" as

$$\pi(y) = \tfrac{1}{5}(1 - 6/462) = 76/385.$$

This is almost 100 times the index for a small nation. Now we can ask the class a good question about applications in general: It is tempting to say a "Big-Five" member is almost 100 times as powerful as a "small" member; in what terms, other than the power index which gives this temptation, would you explain the meaning of "100 times as powerful"?

From 50 to 75 per cent of the work being done in engineering departments of manufacturing plants can be done as well, if not better, by engineering technicians who graduate from two-year technical colleges, rather than from four-year universities.—T. Keith Glennan, President, Case Institute of Technology, Cleveland.—*Taken from* Ed Press Newsletter, *Vol. 18, No. 10, March 4, 1957.*

Edited by Howard Eves, University of Maine, Orono, Maine

Omar Khayyam, mathematician

by D. J. Struik, Massachusetts Institute of Technology, Cambridge, Massachusetts

Introduction

Omar's quatrains, the *Rubáiyát*, are familiar to many of us through the exquisite translation by Edward FitzGerald, often adorned with delicate illustrations. Not all who like the *Rubáiyát*, either because of the verses' melodious English, their philosophy of life often akin to that of Ecclesiastes (or, if one prefers, of Epicurus), or their images of life under Sassanian or Seljuk kings in ancient Persia —not all these admirers are aware that their author, Abu-l-Fath Omar ibn Ibrahim Khayyam, of Nishapur in present North Iran, was also a distinguished philosopher, astronomer, and mathematician. As a philosopher, he was a follower of Aristotle, whom he interpreted with a keen rationalism; as an astronomer, he composed a calendar more accurate than that proposed centuries later by Pope Gregory XIII and now adopted by most people. Omar's mathematics has been brought closer to the English-reading world by the Kasir translation of his *Algebra*, published in 1931, although a French translation by Woepcke has existed since 1851 [1].[1] Recently a Russian translation has appeared, accompanied by translations of some other of Omar's scientific and philosophical writings, so that now we can obtain a somewhat more definite understanding of Omar's position in the history of thought.

[1] Numbers within brackets refer to the Notes and References at the end of the article.

Omar's life is usually placed between 1040 and 1123, but these dates are not certain—let us say that he flourished in the period which witnessed the First Crusade. When FitzGerald, in 1859, published the first edition of the quatrains (an edition which once sold for a penny a copy and now has brought as much as $8000 a copy), he added a preface with a biography of Omar which contains most of what is known about the Persian poet's life. This biography has often been reprinted, and we need not repeat the details. It suffices to recall that Omar, after study at Nishapur, lived a thinker's quiet life near the court of the Seljuk sultans, first at Baghdad, then at Merv, not far from Nishapur and now inside the Soviet Union. He is buried at Nishapur, where the Iranian government, in 1934, erected a marble monument over his tomb.

By birth, Omar was a Tadjik and thus hailed from a region which had figured in antiquity as Bactria, and belonged to a people from which had sprung men of great eminence, such as the Persian poet Firdausi (*c.* 940–1020) and the equally great scientist Ibn Sina (980–1037), also known as Avicenna. The influence of both of these men is visible in Omar's work. Another distinguished writer, a fellow student with Omar at Nishapur, was Nizam-al-Mulk, who became vizir at the Seljuk court and whose book on statesmanship, well known in Iran, was in 1949 republished in a Russian translation. Since the Tadjiks now have their own

republic, one of the constituent republics of the USSR, academic interest in Omar, already existing in Iran, has increased in the Soviet Union. Popular interest, of course, has long existed, for Omar's quatrains, like Firdausi's stanzas, have passed from mouth to mouth throughout the ages among the inhabitants of the regions near the borders of Iran, the USSR, and Afghanistan.

The solution of cubic equations

Our understanding of Omar Khayyam as a thinker has lately been enhanced by the publication, in a Russian translation, of two mathematical, one physical, and five philosophical papers [2], followed by a book sketching Omar's life and works on the basis of a thorough knowledge of all sources presently available [3]. For this work we are indebted to Professors S. B. Moročnik, B. A. Rozenfel'd, and A. P. Juškevič.

Here we shall be primarily concerned with the mathematical contributions of Omar. First of them is the *Algebra*, or, in full title, *On Demonstrations of Problems of Algebra and Almucabala*, in a translation from the same Leiden manuscript in the Arabic language which was used by Woepcke. (The Kasir translation is from a manuscript that was owned by Professor D. E. Smith of Teachers College, Columbia University.) Its interest for us, who already have access to the English translation, lies mainly in the extensive commentary, which supplements that given by Kasir; in it, 1079 is proposed as the year in which the book was finished. Those who have not read Kasir's translation may like to know that Omar, in his *Algebra*, extended the methods explained in Euclid's *Elements* (300 B.C.) for the solution of second-degree equations to those of the third degree, a considerable achievement.

Euclid's method was geometrical; for him a quadratic equation represented a problem on areas, so that the equation which we write $x^2-ax+b=0$ was interpreted as the problem of finding a square (side x) which added to a given area (b) would yield a rectangle of given side a and required side x. Our equation was thus interpreted in the form $x^2+b=ax$, with a and b positive, and all three terms represented areas. Euclid therefore had to consider three types of quadratic equations: (1) $x^2+b=ax$, (2) $x^2+ax=b$, (3) $x^2=ax+b$, with only positive roots recognized, while our type, $x^2+ax+b=0$, does not occur. This ancient theory remained in force, with certain qualifications, until the days of Descartes (1637).

Khayyam's contribution was to extend this theory to cubic equations, in which he therefore had to think in terms of cubes and parallelopipeds, and in which again only positive roots had a meaning. Omar had thus to consider three binomial equations: $x^3=a$, $x^3=ax$, $x^3=ax^2$; nine trinomial equations: $x^3+bx^2=a$, $x^3+a=bx$, etc.; and seven tetranomial equations: $x^3+cx^2+bx=a$, $x^3+cx^2=bx+a$, etc.

Where Euclid could accomplish the actual construction of his unknown segment (our x) with compass and straightedge [5], Omar had to use more advanced methods, and so, following certain special Greek examples (found, for instance, in the works of Archimedes), he obtained his solutions by the intersection of conics. His treatment of cubic equations in the particular sense in which he approached his problem was exhaustive, and he added to it the solution of certain other equations which led to quadratic and cubic equations, such as $1/x^2+2/x=5/4$, $x+2+10/x=20/x^2$, and similar ones.

We used, above, the words, "with certain qualifications." Parallel with the geometrical theory of equations, which was based on the homogeneity of the terms and not easily generalized to equations of degree greater than three (the notion of a space of four dimensions only became familiar in the nineteenth century), there existed a purely arithemetical-algebraic theory. This theory already flourished in the Mesopotamia of the third millennium B.C., and considered equations

as relations among numbers, asking for numbers satisfying the relations with as much precision as was possible or desirable. There was no natural limit at degree three, and in ancient Mesopotamia we even find transcendental equations such as $a^x = b$ (appearing quite naturally in compound interest problems). However, this theory lacked the finesse of Euclid's structure, which was designed, under the influence of Plato and Eudoxus, to deal with both commensurable and incommensurable quantities without concession to rigor.

The arithmetic-algebraic considerations continued to be cultivated through the ages. We meet them in the Hellenistic world despite all exorcisms of the Platonists, and they even reached lofty heights in the work of Diophantus (250 A.D.). This theory also met with success in India and China, and appeared again in Islamic Mesopotamia and Iran. Omar was conscious of this trend, and he refers to the difficulties of finding an algebraic solution of the cubic equation [6]. It was indeed not until the sixteenth century that this solution was found in its generality, and then this happened in Renaissance Italy, where the "Great Art" was first presented to a brave new world by Cardan in the year 1545.

The binomial theorem

Omar seems actually to have studied numerical solutions, if only of equations of the form $x^n = a$ (n a positive integer). He refers in his *Algebra* to a book he had written which dealt with this problem. The manuscript of this book has not (yet?) been found. It seems from the context that Omar must have known the formula for the expansion of the binomial $(a+b)^n$, where n is a positive integer. If so, then he has the prior claim to a result which up to recent years was usually credited to the Lutheran minister Michael Stifel (1544), but which already occurs in the works of Al-Kashi, a Persian living in the entourage of the Khan Ulugh Beg at Samarkand during the first decades of the fifteenth century.

The parallel axiom

The second of Omar's mathematical works now published in Russian, with an ample commentary, is entitled *Commentaries on the Difficulties in the Postulates of Euclid's Elements* [4]. The original Arabic text, from a Leiden manuscript, had already been published at Tehernal in 1936 [5]. The content of this book was partly known to the English reader through an informative paper by D. E. Smith in the *Scripta Mathematica* of 1935; Professor Smith had found some quotations from Omar's text in an ancient manuscript he had bought [6]. Now that we have a chance to inspect the whole book, we see that this *Scripta* article gives a good account of the first section of Omar's book, the section in which he deals with the parallel axiom.

The parallel axiom, which states that through a point not on a line one and only one parallel can be drawn to this line, appears (in a slightly different form) in Euclid's *Elements* as the Fifth Postulate. All through the centuries courageous souls have tried to prove this postulate, that is, to derive it from other more obvious postulates, or else to replace it by some principles which are less complicated. Omar, trying to connect the axiom with the Fourth Postulate, which asserts that all right angles are equal, tried to bridge the gap between the two postulates with the aid of five principles followed by eight propositions. The principles, or stepping stones in his reasoning, are taken, directly or indirectly, from Aristotle, who lived from two to three generations before Euclid. The first principle, for instance, is that quantities can be divided without end, that is, there are no indivisibles. The second principle states that a straight line can be indefinitely prolonged. The next two principles deal with intersecting lines, and the last one is the axiom of Archimedes.

The propositions contain some interesting material. In the first one, Omar proves that if two equal line segments AC and BD are drawn perpendicular to a segment AB, then angle ACD = angle BDC (Fig. 1). In the second one, he proves that if, in this same figure, E is the midpoint of AB, and EG is perpendicular to AB (Fig. 2), then $GD = CG$ and EG is perpendicular to DC. Then, in the third proposition, he comes to the conclusion that angles ACD and BDC are right angles, which indeed establishes the parallel "axiom," from which the other propositions, such as that $CD = AB$, follow.

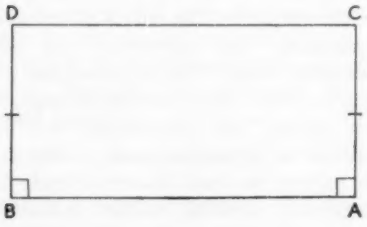


Figure 1

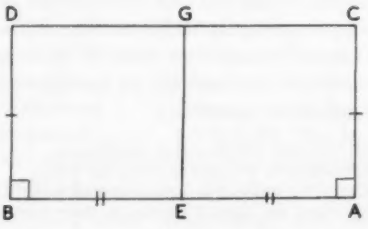


Figure 2

In order to prove the third proposition, Omar finds he must consider three cases according as angles ACD and BDC are each (a) less than a right angle, (b) larger than a right angle, and (c) equal to a right angle. The assumptions (a) and (b) he rejects by constructing $CDFH$ on $ABDC$ (see Fig. 3) by prolonging EG its own length to K and drawing HKF perpendicular to GK to intersect AC and BD prolonged in H and F. Then, by folding the figure first about CD, then about AB, he shows that in both cases (a) and (b)

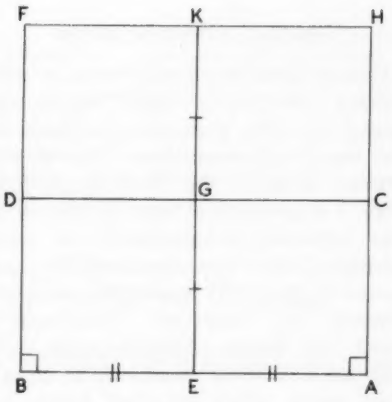


Figure 3

he comes into conflict with his principles (that is, in hidden form, with the Fifth Postulate), so that only case (c) remains.

The interesting aspect of this argument is that we find here, apparently for the first time in history, the three situations later known as the hypothesis of the acute angle (case [a]), that of the obtuse angle (case [b]), and that of the right angle (case [c]). These three situations are now known to lead, respectively, to the non-Euclidean geometry of Bolyai-Lobačevskiĭ, to that of Riemann, and to Euclidean geometry. The three cases were quoted by the Persian mathematician and astronomer Nasir ed-din (1201–74), in whose work Professor Smith discovered them, and Nasir's studies on Euclid became known in 1594 through an Arabic, and in 1651 through a Latin, version. The Latin version was due to the well-known Oxford mathematician John Wallis, whose work was widely read. The terminology, "hypothesis of the acute, obtuse, and right angles," is due to Girolamo Saccheri (pronounced Sak′ keri), a Jesuit mathematician, some of whose ideas, published in a book of 1736 entitled *Euclid Freed of All Blemish*, anticipate later developments in non-Euclidean geometry. From here the terminology passed into our present books on this subject.

The real number concept

Omar's book on Euclid contains still another contribution. After having discussed the Fifth Postulate, he passes to the theory of proportions. This theory appears twice in the *Elements*, first in Book V in geometrical form, developed on line segments independently of commensurability or incommensurability, and second in Book VII in arithmetical form, derived for "numbers" ("arithmoi"), which are always commensurable since they are defined as "composed of units." Two ratios, which we shall denote by a/b and c/d, are defined as equal if, whenever $ma<nb$ then also $mc<nd$, whenever $ma=nb$ then also $mc=nd$, and whenever $ma>nb$ then also $mc>nd$. Here m and n are any numbers (arithmoi). Omar is displeased with this definition, since it is impossible to try all numbers m and n; it also leads to difficulties when ratios are multiplied. Instead of Euclid's definition, Omar prefers to apply his first principle and to define equal ratios by what we might describe as something like a limit process—ratios are equal when they can be expressed by the ratio of integer numbers with as great a degree of accuracy as we like [7]. This amounts to the statement that a ratio can be expressed by numbers with any desired degree of accuracy.

We see that Omar is here on the road to that extension of the number concept which leads to the notion of the real number. For Euclid, with his "arithmoi" and their ratios, only rational numbers had arithmetical meaning. We see the Renaissance mathematicians further wrestling with this subject, especially Stevin (1585), though with him the number concept is still restricted to radicals. The full correspondence of arithmetical and geometrical continuum is reached by Descartes, although the exact definition of the real number had to await the work of Dedekind and Cantor in the nineteenth century.

Summary

Omar thus emerges as a mathematician of considerable penetration, who contributed to the theory of equations, to the understanding of the parallel axiom, to the development of the concept of real number, and probably to the generalization of the binomial theorem. Together with this emergence of Omar as a scientific figure has come a better understanding of his position as a philosopher and a literary figure [8]. The vague medieval figure, outlined by Woepcke in 1851 and by FitzGerald in 1859, has taken on flesh and blood. We now see before us a Persian philosopher of the school of Avicenna, representing the rationalist and anti-theological wing of the Aristotelians (this wing, to which also Averroes belongs, has been called "the Aristotelian left"), a professor of astronomy and mathematics who softens the hard hours of study and struggle by writing mildly ironic-blasphemous verse in the popular style and with a delicate sense of language. And although we know that his own verses in the *Rubáiyát* were quite different from those of his English interpreter both in spirit and in meter, let us sum up by paraphrasing a FitzGerald quatrain:

> Ah, but my Meditations, people say,
> Are idle and will perish with my Clay.
> Yet, if their Fragile Vessel be well steer'd,
> No wind can blow Oblivion in their way.

Notes and references

1. D. S. Kasir, *The Algebra of Omar Khayyam*, Teachers College, Columbia University, New York, 1931, v+126 pp. *L'Algèbre d'Omar Alkhayyami*, publiée et accompagnée d'extraits de manuscrits inédits par F. Woepcke. Paris, Duprat, 1851. There also exists another English translation, by H. J. J. Winter and W. Arafat, *Journal Royal Asiatic Society of Bengali Science* 16 (1950), pp. 27–77.

2. B. A. Rozenfel'd and A. P. Juškevič, "The Mathematical Tracts of Omar Khayyam, with Commentaries" (in Russian), *Istoriko-Matematičeskie Issledovanija* 6 (1953), pp. 1–172.

3. S. B. Moročnik and B. A. Rozenfel'd, *Omar Khayyam, Poet, Thinker, Teacher* (in Russian), Samarkand, Tadjik-gosuzdat, 1957, 208 pp.

4. See R. C. Archibald, "Notes on Omar Khayyam (1050–1122)," *Pi Mu Epsilon Journal* 1 (1953), pp. 351–358.

5. *Discussion of Difficulties of Euclid by Omar Khayyam*, edited with an introduction by Dr. T. Erani, Teheran, 1936.

6. D. E. Smith, "Omar Khayyam and Saccheri," *Scripta Mathematica* 3 (1935), pp. 5–10.

7. On this see also E. B. Plooy, "Euclid's Conception of Ratio and His Definitions of Proportional Magnitudes as Criticized by Arabian Commentators." Thesis, Leiden, 1950, vi+71 pp.

8. On Omar Khayyam as a poet, and an evaluation of the FitzGerald version, see A. J. Arberry, *Omar Khayyam*, Murray, London, 1952, 159 pp.

Omar Khayyam's solution of cubic equations

by Howard Eves, University of Maine, Orono, Maine

Omar Khayyam was the first to solve geometrically, so far as positive roots are concerned, every type of cubic equation. Let us illustrate his procedure for the special type

$$x^3+b^2x+a^3=cx^2,$$

where a, b, c, x are thought of as lengths of line segments. Omar stated this type of cubic rhetorically as "a cube, some sides, and some numbers are equal to some squares." Stated geometrically, the problem of solving the cubic equation is this: Given line segments a, b, c, construct a line segment x such that the above relation among a, b, c, x will hold. The object is to construct x using only straightedge and compasses as far as is possible. A solution using only straightedge and compasses is in general impossible, and at some point of the construction we must be permitted to draw a certain uniquely defined conic section.

A basic construction used several times in the solution of the cubic is that of finding the fourth proportional to three given line segments. This is an old problem whose solution was known to the ancient Greeks. Suppose u, v, w are three given line segments and we desire a line segment x such that $u:v=w:x$. Figure 1 will recall how, with straightedge and compasses, one may construct the desired segment x.

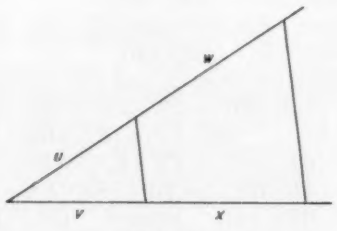


Figure 1

We now follow Omar's geometrical solution of the cubic equation

$$x^3+b^2x+a^3=cx^2.$$

First of all, by the basic construction, find line segment z such that $b:a=a:z$. Then, again by the basic construction, find line segment m such that $b:z=a:m$. We easily find that $m=a^2/b^2$. Now, in Figure 2,

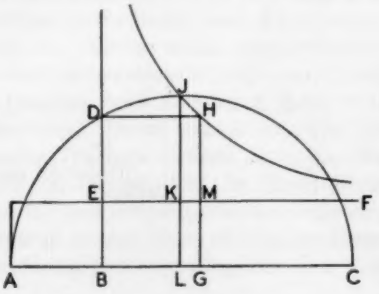


Figure 2

construct $AB=m=a^2/b^2$ and $BC=c$. Draw a semicircle on AC as diameter and let the

perpendicular to AC at B cut it in D. On BD mark off $BE=b$ and through E draw EF parallel to AC. By the basic construction, find G on BC such that $ED:BE=AB:BG$ and complete the rectangle $DBGH$. Through H draw the rectangular hyperbola having EF and ED for asymptotes (that is, the hyperbola through H whose equation with respect to EF and ED as x and y axes is of the form $xy=$ a constant). Let the hyperbola cut the semicircle in J, and let the parallel to DE through J cut EF in K and BC in L. Let GH cut EF in M. Now:

1. Since J and H are on the hyperbola, $(EK)(KJ)=(EM)(MH)$.
2. Since $ED:BE=AB:BG$, we have $(BG)(ED)=(BE)(AB)$.
3. Therefore, from (1) and (2), $(EK)(KJ)=(EM)(MH)=(BG)(ED)=(BE)(AB)$.
4. Now $(BL)(LJ)=(EK)(BE+KJ)=(EK)(BE)+(EK)(KJ)=(EK)(BE)+(AB)(BE)$ (by [3])$=(BE)(EK+AB)=(BE)(AL)$, whence $(BL)^2(LJ)^2=(BE)^2(AL)^2$.
5. But, from elementary geometry, $(LJ)^2=(AL)(LC)$.
6. Therefore, from (4) and (5), $(BE)^2(AL)=(BL)^2(LC)$, or $(BE)^2(BL+AB)=(BL)^2(BC-BL)$.
7. Setting $BE=b$, $AB=a^3/b^2$, $BC=c$ in (6), we obtain $b^2(BL+a^3/b^2)=(BL)^2(c-BL)$.
8. Expanding the last equation in (7), and arranging terms, we find $(BL)^3+b^2(BL)+a^3=c(BL)^2$, and it follows that $BL=x$, a root of the given cubic equation.

It must be admitted that Omar's method is ingenious, and it certainly can interest some high school mathematics students. A point-by-point construction of the hyperbola (utilizing the basic construction) is easy, for if N is any point on EF and if the perpendicular to EF at N cuts the hyperbola in P, then $(EM)(MH)=(EN)(NP)$, whence $EN:EM=MH:NP$, and NP is the fourth proportional to the three given segments EN, EM, MH. In this way a number of points on the hyperbola can be plotted and then the hyperbola sketched in by drawing a smooth curve through the plotted points. The student might be given the numerical cubic $x^3+2x+8=5x^2$. Here $a=2$, $b=\sqrt{2}$, $c=5$. The three roots of this cubic are 2, 4, -1. The student should be able to find the two positive roots by Omar's method; perhaps he can extend the method slightly to find the negative root. Here is the start of an excellent so-called "junior research" project. The student may find Omar's geometrical approach to other types of cubics in J. L. Coolidge, *The Mathematics of Great Amateurs*, Oxford (1950), Chapter II (Omar Khayyam), pp. 19–29. The algebraic solution of cubic equations can be found in any text on the theory of equations.

"Projective geometry is all geometry."—*Cayley.*

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Organizing the mathematics department in the Montebello Junior High School

by Raphael W. Wolfe, Montebello Junior High School, Montebello, California

During the summer of 1957, it was my privilege to receive a grant from the National Science Foundation enabling me to attend an institute for thirty-five teachers of junior high school mathematics. This session, sponsored by State University Teachers College at Oneonta, New York, was one of about ninety such institutes held throughout our country during this summer in an effort to strengthen and advance science and mathematics teaching in the secondary schools of the United States. As it was evident that the National Science Foundation has given much thought to the problems of science and mathematics education, it is also clear that each secondary school should scrutinize carefully its plans and procedures in these vital fields. It is my purpose in this paper to show what steps should be taken to improve the teaching of mathematics in the Montebello Junior High School, Montebello, California. The same reasoning applies in both mathematics and science in all the junior high schools of the Montebello Unified School District. I am basing my recommendations primarily on the needs of the Montebello Junior High School, however, as the situation with which I am most familiar.

It has been my observation, both in the fields of curriculum content and teaching methods, that our present situation leaves much to be desired in the field of mathematics. There is little co-ordination between the activities of our several mathematics teachers. In our school of over 1500 students, our administrators have little time to give to thorough supervision of the general teaching methods of the fifteen or more teachers who teach one or more mathematics classes. And it is impossible for busy administrators to keep up with the trends in rapidly changing modern mathematics. Many of our teachers are not mathematics majors.

In our seventh grade, following somewhat as it does the organization of the core curriculum, some of our mathematics teaching suffers.

It is questionable whether mathematics should be included in a core curriculum. William David Reeve[1] said of the core curriculum: "No one would suggest that one teacher can correlate all of the various subject-matter fields and teach them in the classroom." In an address at State University Teachers College, Oneonta, New York, on July 10, 1957, Dr. Harry Rivlin, who has had wide experience in education and is at present Dean of Teacher Education for the four city colleges in New York,

[1] William David Reeve, *Mathematics for the Secondary School* (New York: Henry Holt and Co., 1954), p. 450.

expressed his opinion in this matter. He said that the core curriculum requires an outstanding type of teacher and even where it is successful, it is usually found advisable to separate the teaching of mathematics and science from the core. These subjects need a more logical method of development. Mathematics teachers are needed who can teach for meaning, and if they are not themselves well grounded in the theory of mathematics they can hardly be expected to get the meanings across to their students. Harold P. Fawcett[2] discussed various aspects of the core curriculum as related to mathematics and concluded after a detailed study that mathematics teaching needs to be organized in ways not likely to be followed in the core curriculum.

In our eighth and ninth grades we have little co-ordination between teachers. Sometimes we even have different teachers teaching the same types of classes with different textbooks and with little subject matter in common. A good teacher who develops fine techniques and a variety of teaching aids has little opportunity to share these with his less versatile colleagues. Each mathematics classroom is practically a little compartment of its own.

I have not set down these criticisms to say that the mathematics teaching in Montebello Junior High School is bad or even below average. I am saying that we do not actually know its quality; there is little opportunity to evaluate procedures under the present conditions. I am sure, though, that there is much room for improvement.

A first step in improving the mathematics instruction in our institution should be the appointment of a head of the mathematics department. He should be relieved of some teaching time and should be paid at least a nominal stipend in addition to his regular teaching salary. He should continue teaching part time, for his understanding of the problems met by his colleagues is thus kept on the "grass roots" level. Consultants from a central office are helpful, but closer touch and more intimate understanding of a fellow teacher are vital for promoting the best teaching.

When such an appointment has been made, the principal should call a meeting of all the mathematics teachers, introduce the head of the department, and make clear with all tact possible that the selection has been made, not with any idea of starting a program of "snoopervision," but rather with the expectation of improving the mathematics teaching in our school. The principal should outline briefly at this meeting what individual teachers will gain by this arrangement, and how he expects teaching to be improved. He should stress some of the ways he expects the department head to be helpful. The latter should then speak, briefly outlining some of his plans and doing all in his power to indicate that this is to be a co-operative undertaking in which it is anticipated that all mathematics teaching will be improved where possible and kept at a high level where it already is above average.

We shall now direct our attention to the program which should be initiated by the newly appointed head of the mathematics department after his introduction. Probably the need for in-service training and evaluation of our teachers is the most critical problem in this institution. As a majority of our teachers are not mathematics majors, there is a great need to acquaint them and stimulate their interest in the trends in modern mathematics. This should be done in department meetings and also by distributing literature describing ideas useful in teaching junior high school mathematics.

Not only should these nonmathematics majors be helped to a greater awareness of the need to teach meanings and newer concepts, but the new and often inexperienced teachers coming into our institution almost every year need special attention.

[2] "Mathematics in the Secondary Schools Today," *The Bulletin of the National Association of Secondary-School Principals, No.* 203, May 1954, pp. 71–80.

Butler and Wren[3] have said, "Large numbers of these [mathematics] teachers come into the schools each year fresh from college, with high enthusiasm for their work, but with little or no experience in teaching and without much firsthand knowledge of the problems connected with teaching secondary-school students. . . . Most young teachers can be greatly helped by the guiding advice and counsel of a sympathetic and experienced supervisor for a period of several years after they enter upon their work. The need for supervision is accentuated by the fact that the majority of secondary-school teachers find it necessary to teach one or more subjects in fields other than those of their major preparation." Pointing out the need of supervision for the experienced teacher as well, they continue, "Extensive and reasonably successful experience is a great asset to the teacher of mathematics, but such experience itself is sometimes not without its potential disadvantages. In particular, the teacher who has been successful in his work may find it easy to get into a rut and to become complacent about his work and oblivious or insensitive to possibilities of further improvement. Thus even teachers of long experience and substantial training may benefit from appropriate supervisory contacts through being stimulated to professional alertness and scholarly advancement."

Thus, since it has been shown that there is need for both new and experienced teachers to have supervision, we should describe briefly some of the methods to be used to implement this program.

Especially in the cases of new teachers, the department head should make frequent visitations. These should, at least at first, be made upon previous notification and should be long enough to be meaningful and profitable. There should be time to note how the class is handled as it begins its work and how the work is carried on. The head of the department should be as unobtrusive as possible. In fact, unless he is asked by the teacher or unless there is an extreme situation, he should not enter into any discussions while the class is in session.

As soon as possible after the visit, at the mutual convenience of the teacher and department head, there should be a conference in which the latter as tactfully as possible discusses the class visited. There should be commendation of at least some phases of the class situation, and criticisms should be constructive. There should be no attempt to fit the teacher into a narrow mold, for different personalities must handle situations in different ways. The department head should be prepared to learn in these situations, too, and should use what he has learned, with credit freely given, in helping other teachers. Each such conference should culminate in a brief written summary by the supervisor, copies to be given to the teacher, filed in the principal's office, and kept by the department head, respectively.

If the situation seems to warrant, it may be advisable for the department head to teach a demonstration lesson. Such a situation is an artificial one, since there is not the usual teacher-pupil relationship on a familiar basis. Another effective way of demonstrating to a new teacher (and, once in a while, to an experienced one) is to arrange to have him visit some classes taught by a superior teacher. The department head can make arrangements for such visitations and help see that they are effective.

Another good way of helping mathematics teachers is the promotion of a workshop. In the field of curriculum, this could be very effective. Teachers and department head, working together, should develop and revise courses of study as general guides. These should not be so inflexible as to give little opportunity for individual ingenuity and special interests of some teachers. In their broad outlines

[3] Charles H. Butler and F. Lynwood Wren, *The Teaching of Secondary Mathematics* (2d ed.; New York: McGraw-Hill Book Co., Inc., 1951), p. 249.

they should help inexperienced teachers and also help prevent lazy experienced teachers from spending too much time on topics they especially like or consider easy, to the detriment of other essential materials. It is necessary for teachers to feel they have a part in determining what shall be taught, and such a co-operative undertaking will work in this direction. A workshop to develop better visual aids and encourage their use could be another project. On the other hand, the department head should not sponsor so many meetings and bring so much pressure on teachers that they may rebel. Meetings should be held as often as is really necessary and only as often as all sincere teachers can really truthfully say they have profited by attending.

The department head should help teachers in evaluating the work of their students and should help teachers evaluate their own effectiveness. He should help direct teachers in their use of tests and should be alert for opportunities to use standardized tests where they will be useful. Such tests should not be used as a club over a teacher whose class does poorly, but rather employed as a means of determining, in such cases, what means the teacher might use to improve his teaching, if it is at fault.

There are many other ways the department head can help make the mathematics teaching more effective. These have been described in various books and articles on teaching, so there is no need here to attempt a complete listing. I will conclude this paper by describing briefly just a few more.

The head of the department, because he is in contact with all the teachers in his department and also has meetings with mathematics educators from other schools, is in a position to learn of worth-while field trips that might not be known by some of the teachers. Where this information can be helpful, he should share it, and should be looking for other opportunities of the like. He should help teachers select and evaluate various materials such as books, audio-visual aids, and others. He should keep at least some departmental records. He should pass on to teachers worth-while articles and books on mathematics. He should carry on some research designed to keep the department alert.

Finally, the head of the mathematics department should be a very helpful person for the over-all school program. He should be one of the persons on whom the principal can depend for help in some administrative areas. The reports of class visitations, previously described, should be very helpful to the principal in his task of rating and recommending teachers. The department head should help in the sectioning of classes so that the slow learners are given attention apart from the average and above average students. He should coordinate the ordering and care of materials, such as flannel boards, micrometers, abacuses, charts, and texts. He should help develop closer co-ordination between junior- and senior-high-school mathematics departments.

The organization of a mathematics department at Montebello Junior High School will not immediately guarantee that the quality of all teaching will improve, but it will help focus attention on problems peculiar to the teaching of this subject. There should be a gradual improvement in the situation. Since the importance of mathematics in our civilization today is receiving so much attention, it is hoped that the plan outlined in this paper can be put into effect as soon as possible.

Bibliography

Butler, Charles H., and Wren, F. Lynwood, *The Teaching of Secondary Mathematics* (2d ed.). New York: McGraw-Hill Book Co., Inc., 1951. Pp. 54–68, 248–265.

Education for the Talented in Mathematics and Science. (Bulletin of the United States Department of Health, Education and Welfare No. 15.) Washington, D.C.: Government Printing Office, 1952.

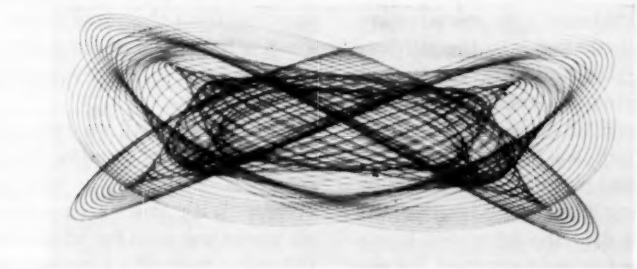
HANNA, PAUL, *Opportunities for the Use of Arithmetic in an Activity Program.* (10th Yearbook of the National Council of Teachers of Mathematics.) Washington, D.C.: The National Council of Teachers of Mathematics, 1935.

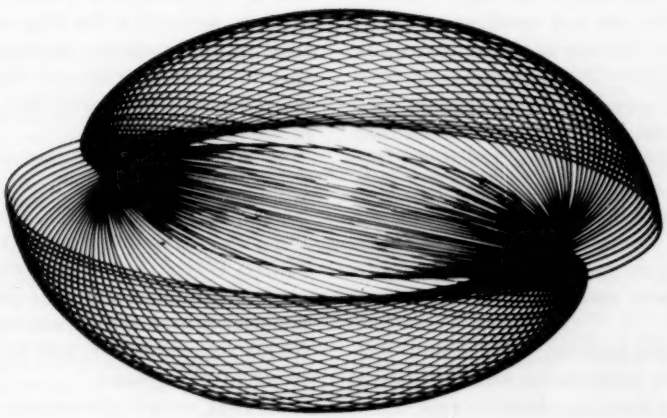
"Mathematics in the Secondary Schools Today," *The Bulletin of the National Association of Secondary-School Principals, No.* 203 (May 1954).

REEVE, WILLIAM DAVID, *Mathematics for the Secondary School.* New York: Henry Holt and Co., 1954. Pp. 121–122, 450.

VAN ENGEN, H., "Arithmetic in the Junior-Senior High School," *The Teaching of Arithmetic* (50th Yearbook of the National Society for the Study of Education, Part II), 1952. Chap. vi.

Pendulum patterns





—George T. Hillman, Park Ridge, Illinois

A column of unofficial comment

Some comment on accelerated mathematics

by Daymond J. Aiken, Lockport Township High School, Lockport, Illinois

There are two definite trends in secondary mathematics today: the introduction of modern mathematics into the curriculum, and the introduction of accelerated programs to gifted students. It is my belief that we should take a closer look at the second trend.

The accelerated program means moving faster and going farther—extending the secondary level into the advanced topics area which was formerly reserved for the college student. Such a program, if properly handled, is surely good for the gifted and is not to be belittled in any sense. In this regard, however, there is one particularly important question that needs to be answered. How about the many capable students who are not rated as gifted in high school? Are we to neglect these? In my opinion, they too can profit much from the study of advanced topics. In fact, on a relative basis, I am convinced that they will profit more than the gifted group. The gifted can adjust to the college program, while many of the less gifted get lost. Consequently, they either drop out or decide that a course requiring mathematics is not for them and change their field of study.

During the past ten years, I have participated in projects in which advanced topics have been introduced to high school students. The first group was in 1947. It began with a solid geometry class. Toward the end of the semester it was felt that the class had had all the solid geometry necessary. So, for a change of menu, they were given a little over three weeks to explore advanced topics (mostly introductory calculus).

The following fall we were astounded by the good reports coming back from the members of this group who had entered college. With this encouragement, about six weeks was used for advanced topics the following year. We acquired this time by speeding up and abbreviating some of the work in trigonometry. Again, good reports came back from our college freshmen.

From 1947 to 1952, we continued this method of sandwiching in advanced topics, that is, we speeded up and abbreviated as much as we could in the trigonometry and solid geometry classes. Always, it seemed that the more time allowed for these topics, the more enthusiastic were the reports in the fall.

In 1952, we began introducing the advanced topics in the junior mathematics classes. In the senior year, this same material was covered again. Big dividends resulted. When the topics were taken up again in the senior year, the ideas had had one year to be absorbed, and the result was better comprehension and understanding of the ideas involved.

In 1954, we introduced into our curriculum what we call Senior Mathematics. This consists of trigonometry and solid geometry for the first semester, and calcu-

lus and analytical geometry for the second semester. The solid geometry was very abbreviated (about four or five weeks: one week for three-dimensional drawings and review of geometric terms and principles, one week for developing some of the basic theorems, then two weeks for solving a battery of typical solid geometry problems).

It is, I suppose, an easier teaching situation to have the gifted students in one class, but even this situation will not produce a homogeneous group—thank goodness! Individual differences will be present regardless of the grouping, and the teacher will have to adapt his plans to meet these differences. Why not eliminate the many problems involved in selecting the gifted students by giving the advanced topics to all students enrolled in the third- and fourth-year mathematics classes? After all, these are selected students, selected by the very fact that they have been admitted to third- and fourth-year classes in mathematics.

It is not difficult to take care of individual differences in classes of this sort. The gifted students can be given plenty to challenge them to any extent desired. They can be given a larger number and/or more difficult problems to solve. They can explore supplementary books. They can assist in teaching the slower students, and so forth.

In our mathematics program, we have not participated in the advanced placement tests given by the College Entrance Examination Board. We have merely aimed at preparing as many students as possible for college mathematics. I am sure, though, that a number of members in each class we have had would have rated high on the examinations for advanced credit. From our viewpoint, the accelerated group method is wrong when it provides only for the gifted student. It does not give those who are less gifted, slower in maturing, or lacking motivation, a chance at the advanced topics. We have seen several of our average students go on to college to become B and A students. If they had not had the advanced topics in high school, we are sure they would have been swamped in college mathematics.

Most important is the avoidance of rifts and controversy in the ranks of education itself. A damaging and in my opinion useless controversy has raged about the manner in which the high school system of the country fails to provide adequate preparation for college and university. It is quite true that every college without exception has to take care of some unfinished high school business. It is quite true that the problems of numbers, costs, and quality of instruction are made more difficult by this alleged defect or delinquency in secondary education. Somewhere we manage to lose at least a whole year and maybe more in the preparation of students who go on to college. But it is also true that in the system of secondary education in this country, preparation for college is only one of many educational functions. It is not even the most important. The pattern of secondary education is set for our generation. The meaning of this is clear. The responsibility of dealing with the deficiencies of the high school graduate remains with the colleges and universities. It cannot be lessened by quarreling with the high schools, nor met by the premature and intemperate severity with which insufficient preparation is often punished in the first two years of college. If it is true that we are approaching a serious shortage in skilled and highly qualified manpower in the nation, then we cannot afford to flush out potential talent before it has had time to prove itself.—*Taken from an article by Cornelis W. deKiewiet, "Action Under Way," American Council of Eduation.*

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

An Adventure in Geometry, Anthony Ravielli. New York, The Viking Press, 1957. Cloth, 117 pp., $3.00.

This book of sketches of shapes of natural objects rendered in two colors should catch the eye of both student and teacher. The emphasis is upon geometric forms. Recognition of familiar figures is mildly encouraged. An introduction to geometric terms and concepts is attempted in the running commentary.

The book is composed primarily of picture sketches, well over half of the space being so used. Each sketch is intended to have geometric significance. The association with geometric concepts is verbally specified in most cases; subtle in many cases, misleading in a few. The reviewer finds most of the pictures pretty but cluttered. The latter feeling may reflect on overlong exposure to the austerity of most figures commonly associated with geometry. However, it must be admitted that many pictures include much more than meets the untrained eye. There is some question as to how much the untrained eye will see.

The commentary covers a wide range of geometric concepts—nearly all simplified. This reviewer was disturbed by a few evidences of misleading oversimplifications. Most of these disturbances are commonplace, and other readers may not find them bothersome. Two examples will be cited. "An angle is the figure formed by two lines drawn from the same point" (p. 42) confuses lines with rays or line segments. "A straight line moving perpendicular to its own direction produces a plane surface" (p. 45) assumes that there is a unique perpendicular direction. Actually one can consider a wheel with each of its spokes perpendicular to its axle. Furthermore, the perpendicularity was superfluous.

Such criticisms are small relative to the very wide range of topics covered. The book may be divided crudely into quarters concerned with (1) designs found in nature, (2) basic figures of plane geometry, (3) basic figures of solid geometry, and (4) such advanced topics as cycloids, spirals, "the shape of sound," projections, shapes which "affect our feelings."

The goals and subtlety of the presentation are connoted by the concluding remarks, "Perhaps in turning the pages of this book, in sharing with us our 'adventure in geometry,' you too have caught a glimpse of that bare and austere beauty that lies beneath the variegated trappings of the world. The beauty of geometry is a hint that there is an order and plan of things too vast for our minds to comprehend."

The book is recommended as a curiosity, an eye catcher. It is not a textbook. It would be a good book for students to explore before, during, or after taking a course in geometry. The book is unique in its simplified representation of geometric concepts as they are related to objects and phenomena of nature. As such, one can reasonably hope that some students will be stimulated to study more geometry and that many will feel that geometry is related to their lives.—*Bruce E. Meserve, New Jersey State Teachers College, Montclair, New Jersey.*

Analytical Conics, Barry Spain. New York, Pergamon Press, 1957. Cloth, ix+145 pp., $5.00.

This British text differs from the typical American analytic geometry text in a number of respects.

Its treatment of conics is much more comprehensive. The conics are first treated in their standard forms. The metric transformations and the invariants of the quadratic form are followed by a study of the general second-degree equation. Up to this point, about midway of the text, its distinctive nature is in terms of the treatment of the mathematical content rather than the content itself.

The second half of the text differs radically from its American counterpart. Throughout the text, the central theme is the conic. There is no consideration of algebraic curves in general or of transcendental curves. Polar co-ordinates are not used. Parametric equations are used only incidental to studying the conics in standard form. On the other hand, such concepts as homogeneous co-ordinates, the point and line at infinity, harmonic range and pencil, homographic correspondence (the projective transformation), line co-ordinates, duality, and generalized homogeneous co-ordinates are introduced and utilized in the further study of the properties of the conics.

The book contains no worked-out illustrative examples. Exercises for the reader to work are labeled "examples." An appendix contains a discussion of the solution of most of them. Determinants are used rather extensively throughout the text. It is assumed that the reader is familiar with hyperbolic functions. In the first half of the text, one finds some interesting variations from the usual procedure. The usual parabola, ellipse, hyperbola order is not followed. The ellipse is studied first. It is defined by means of a constant which, in terms of

eccentricity, turns out to be $\sqrt{1-e^2}$. The hyperbola is next introduced by replacing K^2 with $-K^2$. Finally, the parabola is considered as the limiting case for the ellipse and hyperbola. The reviewer found this an interesting treatment.

The text is very concise and compact. The average undergraduate will find it difficult to read. Some of the theory usually developed in the text is left for the reader. For example, the development of the point-slope, slope-intercept, two-point, and intercept forms of the straight line are posed as exercises. The current prevalent criticisms of analytic geometry as a preparation for the calculus are as valid criticisms of this book as they are of the typical American text. The author has assumed the reader has considerably more mathematical maturity than is possessed by the typical college freshman. No attempt is made to develop the basic underlying concept of analytic geometry, the one-to-one correspondence between the ordered pairs of a function and the points of its graph.

For a mature study of the geometry of the conic, this book is excellent. Emphasis on the concept of the invariants of a transformation, use of the extended plane, homogeneous co-ordinates, duality, and line co-ordinates should make of the book a valuable introduction to advanced geometry. For all but the best students, it would be better if fuller explanations of new concepts were given.—*J. Houston Banks, George Peabody College for Teachers, Nashville, Tennessee.*

College Algebra, Irwin K. Feinstein and Kenneth H. Murphy. Ames, Iowa, Littlefield, Adams and Company, 1957. Paper, xi+362 pp., $1.75.

This book is a member of the "Littlefield College Outlines." It is not intended for use as a classroom text, but rather to "complement and assist . . . the classroom teacher and textbook." Essential elements of the theory are explained briefly and then illustrated by several explicit examples. There are no problems for solution.

The authors have not only assembled a multitude of carefully worked problems, but have also attempted to present a self-contained exposition of all of the important topics of intermediate and college algebra. A plethora of definitions is given. Some definitions are even repeated, as in the case of equivalent equations (p. 103 and p. 303). The authors have extended themselves to include definitions of not only all of the algebraic terms in common usage but also many uncommon terms, such as strong and weak inequalities (p. 205), defective and redundant equations (p. 108), and alternation and composition with reference to proportions (p. 175). Most of the definitions are clearly stated, although the definition of finite and infinite sequences (p. 220) is inadequate.

The general exposition is concise and direct. There seems to be a definite emphasis on outlining procedures for algebraic operations, rather than on developing a fundamental understanding of the process involved. Techniques are sometimes presented with little thought to motivating understandings, as for example, in dealing with linear systems (p. 117 and p. 119). Some theorems are proved quite carefully, but many others are merely stated axiomatically and then illustrated. A wise choice has not always been made as to what theorems should be proved. For example, the theorem on rational roots of a polynomial equation with integral coefficients (p. 309) is not proved, although its proof is simple and very necessary to promote understanding. Time and space have been taken, however, to prove the corollary to this theorem dealing with integral roots (p. 310). In the statement of Descartes' rule of signs, no attempt has been made to explain the meaning of the phrase "variations in sign." To the average independent reader, the remark following the statement of this theorem would not be evident (p. 325). The exposition at times degenerates to giving directions. At other times, it is quite clear and stimulating, as in Chapter 13 in which inequalities are considered.

The outstanding feature of this book is the collection of rather well-chosen and carefully worked out illustrative examples. For the most part, the problems are adequately graded as to difficulty and sufficiently varied so as to demonstrate quite effectively the algebraic process under discussion. Some topics have been illustrated too lightly, as in the case of functional notation (p. 84). Some of the examples chosen are unduly complicated, as in the case of complex fractions (p. 60), where all of the illustrative examples are so involved that they mask the principles which should be stressed. Generally, however, this is not the case.

It is unfortunate that problem solving has not been given more emphasis at appropriate stages. A final chapter contains the only mention of statement problems, and consists of a detailed analysis of twenty-one random problems. Little help is provided to assist the reader in developing a variety of tactics in attacking such problems. The use of sketches and diagrams is completely neglected. Verbal methods alone are employed.

Generally speaking, the format of the book is good. There are awkward spots, however, as at the bottom of page 10 where illustrations of one principle are crowded together with statements of a different principle. Mistakes and misprints appear to be very scarce.

The reviewer is of the opinion that this book would be very helpful to someone who has had previous exposure to algebra and is interested in renewing familiarity with it. It has limited value as a supplementary source for a student currently taking a course in College Algebra. There is always the risk that such students will attempt to use it merely as a "crutch," employing the illustrative problems as patterns with little thought given to understanding. For in-

dependent study it would not be recommended since many necessary theorems are not proved, discussions are very brief, and no sets of exercises are provided.—*David D. Strebe, State Teachers College at Oswego, Oswego, New York.*

Niels Henrik Abel—Mathematician Extraordinary, Oystein Ore. Minneapolis, University of Minnesota Press, 1957. Cloth, 277 pp., $5.75.

On August 5, 1802, there was born in Finnoy, a little island off the coast of Norway, a child, Niels Henrik Abel, who during his adolescence and young manhood was destined to become a genius in the field of mathematics. His death, April 6, 1829, of tuberculosis was untimely, as he had shown himself to be a productive mathematician of great promise.

Professor Oystein Ore has written an outstanding biography, the first in English, of Niels Henrik Abel. It begins with Abel's parentage and continues through years dominated by poverty, indebtedness, loneliness, and illness. The biography moves with captivating interest through Abel's twenty-six years and concludes with the account of a two-volume work containing the manuscripts and articles written by Abel, which appeared ten years after his death,

This biography has many interesting and worth-while features: the format is well done; the type is of good size; the paper is of excellent quality. Included in the book are sixteen pages of illustrations including an illustration of the only authentic portrait of Niels Henrik Abel, painted in 1826. Mr. Ore has made extensive use of the many letters written and received by Abel. Many of these letters are used in part or entirety, and have the effect of carrying the story with a close chronological sequence.

It is extremely difficult to compress into 270 pages the full life of Niels Henrik Abel. Here is a mathematician who started new mathematical fields. Even as a youth at the university, Abel's thinking was far beyond that of men his senior by many years.

Readers of this book will be educated as well as entertained. The mathematical contributions and experiences of Crelle, Bessel, Cauchy, Jacobi, Fourier, Legendre, and others become more clear as a result of this reading. This book can be read and enjoyed by the high school or college student. It is not a book treating mathematics formally, but it is the everyday life of a great mathematician whose humble and lovable ways endeared him to his colleagues. For the person interested in mathematics, this book shows problems that confronted individuals who gave us much of our modern mathematics.

Professor Ore pays high tribute to Abel when he states, "Mathematicians, in their characteristic manner, have erected many monuments to Abel more durable than bronze. The custom prevails of marking new results and great ideas by the name of their originator. Today, anyone who reads advanced mathematical texts will find Abel's name perpetuated in numerous branches of his science: Abelian theorems in abundance, Abelian integrals, Abelian equations, Abelian groups, and Abelian formulas. In short, there are few mathematicians whose names are associated with so many concepts of modern mathematics. Niels Henrik would have been greatly surprised at his own importance."—*Samuel J. Jasper, Ohio University, Athens, Ohio.*

Preparatory Mathematics I, Outlines and Exercises, Joel S. Georges, Theodore S. Sunko, Milton D. Eulenberg, and Harold M. Piety. Ann Arbor, J. W. Edwards, Publishers, 1957. Paper, 208 pp., $4.00.

This volume is subtitled "Outlines and Exercises." If this book is used as a reference book to hand to a few weak students, it may prove valuable. Unfortunately, it is presented as a possible textbook, and will probably find use as a textbook—hence, we are forced to evaluate it as a textbook. In such a context, it raises five important questions that are in some measure symptomatic of the ills of American mathematics education in general.

1. *Arithmetic without meaning:* We have learned to make excellent presentations of arithmetic so as to give meaning to every operation. For example, dividing up a rectangle can give meaning to $\frac{1}{2}\times\frac{1}{3}$, and show why the sensible answer *ought* to be $\frac{1}{6}$. *Preparatory Mathematics* makes little use of this *meaningful* approach to arithmetic.

2. *Structure:* "Meaningful arithmetic" such as we have just mentioned might better be called *intuitive arithmetic.* It is arithmetic developed inductively from ordinary reality experience. There is a second approach to arithmetic, namely, deduction from initial axioms, using basic undefined elements. In 1957, the college graduate needs a clear understanding of both methods. *Preparatory Mathematics* presents neither, but substitutes instead a set of rules and rote procedures.

3. *Age level:* According to the Preface, *Preparatory Mathematics* is intended for college freshmen. This is incredible. The contents—addition, multiplication, division, percentage, simple word problems—should have been learned in the elementary grades and in junior high school. In an age of quantum mechanics, nuclear energy, jet airplanes, and man-made satellites, how can we possibly account for a student who matriculates at college without having learned elementary arithmetic?

Even if he does appear at the college portals, how can we justify teaching him a "college" course that would be substandard even on the elementary school level?

4. *New topics:* One way to enliven remedial college work is to introduce new topics, and to practice fractions while learning about statistics or operations research. *Preparatory Mathematics*

contains virtually no trace of modern material, except for a few brief remarks on statistics. The concept of function is not only not explained—it is never even mentioned!

5. *Traditional obscurity:* It is something of an established tradition that obscure statements—or even out-and-out nonsense—are part and parcel of mathematics texts. Modern writers have attempted to give this tradition an appropriate burial. However, *Preparatory Mathematics* still contains some of the old clichés. A few examples:

"A formula may be solved for any letter in it." (P. 81.)

"The locus of a point is the set of all points which satisfy given conditions, and no others." (P. 49.)

"A set of points may mean a single point or a finite number of isolated points; or it may mean an infinite number of points such as all the points on a line, or all the points in a portion of a plane." (P. 49.) [Omits the null set.]

"When a point moves in only one direction, its path is a straight line." (P. 38.) [Offered as the definition of "straight line."]

When one considers H-bombs, color TV, and Sputnik—or the work of the University of Illinois group, the CEEB Commission on Mathematics, and so on—the least that can be said is that *Preparatory Mathematics* is an instance of walking when we need to be running.—*Robert Davis, Syracuse University, Syracuse, New York.*

Trigonometry, Rolland R. Smith and Paul P. Hanson. Yonkers, World Book Company, 1957. Cloth, x+470 pp., $3.72.

This is a good book for the traditional secondary school course in trigonometry. Color is used throughout the text as a valuable teaching aid, and the format of each page is most attractive. The text features a wide variety of practical problems from high school physics, aviation, military operations, and three-dimensional situations.

A unique feature of the text is pointed out by the authors in the preface to the book. It would seem wise to let their own words state the case. "After considering the varied requirements of teachers and students, we decided to present the material in chapters, as is traditionally done, but to sub-divide each chapter into a number of lesson units. Each lesson presents a distinct topic that, in most cases, represents one day's work for an average class." Thus the teacher has a built-in lesson plan available from the start, and this reviewer must admit that the authors have adhered to this plan most religiously.

Another pleasing feature is the "Preparatory Exercises" inserted at the beginning of each lesson. These are wisely designed to review previously developed concepts which will be encountered in each specific lesson, and thus, intelligent motivation for each topic is provided. Each lesson ends with a carefully planned set of review exercises.

The authors have gone to some lengths to indicate a minimum course in trigonometry and yet, within the confines of one book, they have attempted to provide more topics for the better-than-average student.

The vast array of illustrative problems, coupled with the careful exposition found throughout the book, would make it possible, in the opinion of the reviewer, for some students to do the course with little assistance from the teacher. The book comes equipped with the customary four-place logarithm and trigonometry tables.

In addition to the features already mentioned above, added strong points are the lessons on graphs of the trigonometric functions, complex numbers, and slide ruler. There is also an interesting lesson on "dead reckoning" and "plane sailing" fairly early in the text.

The graphs of the six trigonometric functions are carefully and artistically presented with color featured throughout the presentation. Graphs of the type $y=a \sin x$, $y=\sin ax$, $y=b \sin ax$ are carefully presented, as are the topics of periodicity and amplitude. The trigonometry of number is briefly mentioned and left for a future course. This chapter is better than those appearing in most high school texts.

The complex number is equally well treated, with attention paid both to the $a+bi$ and $r(\cos\theta+i\sin\theta)$ forms. De Moivre's Theorem is stated and the usual problems done. Reference is made to the fact that $\sin\theta=(e^{i\theta}-e^{-i\theta})/2i$ and $\cos\theta=(e^{i\theta}+e^{-i\theta})/2$, with a brief discussion of this fact on a purely intuitive basis.

The lessons on the slide ruler are the best self-teaching units this reviewer has seen. He would back up the plea of the authors that these lessons be included even in the minimum course.

This reviewer feels that the major weakness of the text is its length. There are eighty lessons suggested, if all the optional material is included. Some fifteen lessons, totaling approximately one hundred pages, are devoted to the right triangle and related topics. Triangle-solving and its application to various practical situations are obviously featured, almost to the point of being exhaustive, as well as exhausting. It is only fair to state that there is merit in doing this, if the primary target of a course in trigonometry is surveying and navigation. This writer happens to feel that the better features of the text would have been accentuated more vividly had the triangle-solving sections been condensed considerably. Entirely too much time was spent in solving triangles, first by natural functions, then by logarithms, and finally by the use of the slide ruler.

In their laudable attempt to make the book self-teaching, the authors may have erred in the direction of actually doing too much for the student and thus curtailed the opportunity for discovery on the part of the pupil.

All in all, however, if the lessons are chosen judiciously, this book will serve as an excellent text for a reasonably contemporary course in high school trigonometry.—*Edwin C. Douglas, Taft School, Watertown, Connecticut.*

BOOKLET

Arithmetic, Algebra, Logarithms, and Slide Rule with Practical Applications, Robert L. Erickson, Lakeland College, Plymouth, Wisconsin. Booklet with four-place logarithm tables, 104 pp., $1.25 (20% discount in lots of five or more).

This booklet is intended for use as a textbook at either the high-school or college level. It presents a review of topics and skills from arithmetic and algebra before taking up the topics of logarithms and the slide rule. Answers to odd-numbered exercises are found at the front of the booklet.

From a pedagogical point of view, there are several weaknesses in the booklet. The review of arithmetic and algebra is presented rather mechanically and contains some statements that are vague or misleading. A few examples follow.

1. On page 1, the author describes using pebbles of different colors to explain our decimal system of notation. He concludes, "Thus with only 40 pebbles (10 each of white, red, black, and green color), a count of 11,110 can be reckoned." A careful student working this out will find that the author should have said 10,999 with an excess of 1 white, 1 red, and 1 black pebble which cannot be used.

2. Exercise 8-A-11, on page 13, and the answer provided suggest that the principle, "that the product is always the same irrespective of the order in which factors are taken," is known as the *associative law*. This is vague and possibly misleading. If we consider only two factors, then this principle is known as the *commutative law*. If we consider three or more factors, the statement given by the author involves both the associative and commutative laws.

3. The author identifies *ratio* with *fraction*—a very common misconception.

4. The axioms given on page 29 are phrased in the traditional form, "If equals be added to equals, the results are equal." Such form is misleading in the opinion of the reviewer. A much better form is, "For every real number c, $a=b$ implies $a+c=b+c$."

5. The meaning of negative exponents is developed by considering $10^2 \div 10^4$ with the use of the rule for subtracting exponents when dividing a power by a power of the same base. This is not good logic and does not call attention to the fact that a negative power of a nonzero base is *defined* to be the reciprocal of the base raised to that power which is the absolute value of the negative exponent.

6. On page 55, students are led to believe that $\sqrt{4}$ is defined to be ± 2, rather than the conventional meaning.

From a physical point of view, the printing of the answers and the supplementary exercises on slide rule is not attractive. Actually, the process for these parts of the booklet is some kind of offset duplication.

In the opinion of the reviewer, the best part of this booklet is the material dealing with logarithms and the slide rule. Here again the author is not as careful as he ought to be in making statements, but on the whole, this section is pretty good. The conclusion of the reviewer is that the best use of the booklet would be with the high school or college students who do not need the review of arithmetic and algebra which is provided.—*Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa.*

> . . . the woof and warp of all thought and all research is symbols, and the life of thought and science is the life inherent in symbols; so it is wrong to say that a good language is important to good thought, merely; for it is the essence of it.—*Charles Sanders Peirce.*

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

How a mathematics test is born

by Marion Epstein and Sheldon Myers, Educational Testing Service, Princeton, New Jersey

Any mathematics test should be conceived, constructed, and used for a specific purpose. Among the different purposes to be served by tests are diagnosis, selection, placement or sectioning, and appraisal of status or growth. A test which is particularly suited for one of these purposes will often not be suitable for another purpose. A test which attempts to serve many purposes is likely not to serve any of them well. Teachers who intend to construct or use a test in mathematics should spend considerable time in thinking through the uses to be made of the test. The first step, then, in the process of creating a mathematics test is to decide the purpose to be served by the test.

Using resource committees

At the Educational Testing Service, the determination and assessment of testing needs in education is a continuous process in which the profession at large participates. When a new testing need is identified, the careful and laborious process of developing a new testing instrument begins. Usually one of the early steps in this procedure is the selection of a committee of teachers and resource people, representative of different types of schools, different sections of the country, and different points of view on curriculum. This committee will study the problem, formulate the basic principles underlying the test, and make recommendations about the nature and scope of the new testing instrument.

A typical example of such a group was the mathematics committee for the Sequential Tests of Educational Progress (STEP tests) of the Co-operative Test Division of Educational Testing Service. This committee was asked to develop a plan for measuring growth in mathematics for general education from grades four to fourteen. Their first task lay in defining such a curriculum and in identifying those basic concepts in which growth was to be tested. Next, the nature and content of the tests were spelled out.

It was only after this preliminary work that the writing of questions began. When over one thousand new questions had been prepared and reviewed, the mathematics section of the Educational Testing Service assembled sixteen experimental pretests which were tried out on appropriate groups of students. The results were analyzed statistically, questions were revised or rejected, and eight tentative final forms were assembled. These tentative forms were then submitted to the committee and to other groups of teachers for review and comment before final revisions were made and the tests were ready for printing. Through the use of a broadly based committee and thorough review by teachers representing a variety of schools and cur-

ricular approaches, it became possible to produce tests which would be useful to a wide range of schools. During this long development process the original purpose of the tests was kept clearly in focus.

The Educational Testing Service works in somewhat different ways with other committees, such as the Committee of Mathematics Examiners of the College Entrance Examination Board. This committee maintains supervision of an on-going mathematics testing program, in the form of the Intermediate and Advanced Mathematics Examinations. The regular review of test rationales for an on-going program in the light of changing curricular emphases can sometimes be more difficult than developing a new program. One reason for this is that scores should be kept comparable from year to year. In every case, however, the formulation of an up-to-date and clear statement of the measurement problem and the purpose of the test is the central core of all test work.

Constructing a test

The major steps in the construction of a test, after the initial planning and the preparation of a statement on the purpose and rationale, are as follows: (1) the establishment of a detailed set of test specifications based on the rationale (including content coverage, number and type of questions, timing, and level of difficulty), (2) the writing, review, and pretesting of new test questions, and (3) the assembly and review of the final test.

Much additional work may also be done on any one testing instrument. A factor analysis might be done to determine statistically what components are being measured by the test. A factor analysis might also be done to determine what contribution the test makes to a test battery. Validity studies might be made to find out how well the test scores correlate with some other criterion, such as college grades. If the test is published, then norming of the scores for selected populations might be done. If scores are to be compared with scores on tests of previous years, such as in the College Entrance Examinations, then statistical work is done to convert the scores to the regular College Board scale of scores.

We shall now discuss steps two and three above.

Writing the questions

In the writing of questions, there are certain aims that are always kept in mind. Each question should be designed to test a single, well-defined concept, and if it is written to test that concept, it should not be workable by an easier, alternative technique. Answer choices should be based on plausible, but incorrect, methods and misconceptions common among students, not on arithmetic errors unless skill in an arithmetic process is the thing being tested. Time-consuming calculations not germane to the concept being tested are to be avoided, and fairness to the candidate should be given constant consideration by the test constructor. Problems whose solution depends on a trick are always excluded, and care is exerted to measure only those concepts which are considered appropriate for the group being tested, according to the original test plan and specifications. For example, in an academic aptitude test, we are interested in measuring insightful behavior in a mathematical context familiar to all students. In an achievement test on the other hand, we are interested in testing a wide sampling of those basic concepts and problem-solving skills that constitute the subject matter to be covered. The content of the graded school tests of the Co-operative Test Division is adjusted to the appropriate grade levels, and the vocabulary used in the reading matter in these tests is controlled by comparing with word-frequency counts.

Contrary to a prevailing misconception about objective mathematics achievement tests, such tests are not inherently limited to the measurement of mere recall of factual material or of mechanical manipulations of symbols. On the contrary, the full

gamut of concepts and understandings of a good mathematics course can be sampled and tested by ingenious item writers.

Reviewing the questions and the test

One of the most important aspects of the test construction process is review—review of the individual test questions for possible ambiguities or mathematical difficulties by several different people competent in mathematics, and review of the test as a whole after it is assembled. A test is like a symphony in the sense of having a unity and an internal structure and design. A review of a test takes into account the internal structure with respect to variety, coverage, and distribution of the difficulty of questions. Often an outside committee will also examine and help revise questions or suggest substitution of different questions for some in the test. This careful scrutiny at every stage makes the failure of a question due to faulty wording a rare event in mathematics test work of this type.

The main purpose in pretesting is to aid in the selection of questions for final tests by determining the difficulty level and discriminating ability of each question and by providing the basis for rejection or revision of questions which test out as too easy or too hard for the group being tested or which do not function effectively for some other reason, such as possible ambiguity.

Usually a pretest is given to a large group of students whose background parallels that of students who are to take the the final test. A random sample of the pretest answer sheets is used for analysis of each individual question with the aid of IBM machines. For each question, two measures are computed—an index of difficulty and an index of discrimination. The index of difficulty is based on the per cent of students who answered the question correctly. The index of discrimination is based on how well the question discriminates between the students who do well on the test as a whole as compared with those who do less well. In addition to computing the two indices, the IBM machines record the number of high and low students who chose each answer choice on each question.

Final test assembly

In putting together a final test from pretested material, the over-all picture must be kept constantly in view, and the pretest statistics are used as an aid to, not a substitute for, professional judgment. For example, if a test should have a very high mean index of discrimination, this would suggest that the test as a whole is homogeneous in content and is not testing a wide variety of concepts and skills. Since most mathematics tests are designed to have a broad range of content for variations in curricula and differing levels of ability, the average index of discrimination is kept from being too high. On the other hand, this index is kept from being too low, since then the scores would become very difficult to interpret. Occasionally, in spite of low indices of discrimination, questions will be included in a test because professional judgment indicates that they are testing a worthwhile concept. By controlling the indices of difficulty and discrimination on a final test, it is possible to achieve desired reliabilities within particular score regions or to produce different score distributions for given groups.

The most important characteristic of any measuring instrument is its validity, that is, whether a test is measuring what it is supposed to measure. This is not to be confused with reliability, or how consistently it is doing the measuring. Reliability is prerequisite to validity, but not the reverse. Probably the best means at present for assuring the validity of the measuring instrument is through careful initial planning which controls the process of test construction or test revision. Pretesting alone will not guarantee that a test will measure what it is supposed to measure. Validity is largely determined by the adequacy of the original planning, the implementation of these plans in terms of test specifications,

and the establishment of a sufficiently large pool of pretested questions.

From this discussion it should be apparent that standardized mathematics tests are scientific instruments designed for a special purpose. They will give the best results when teachers use these tests for the precise purpose intended and when the results are interpreted with full understanding of their meaning and limitations. The long chain of events from the initial recognition of the need for a particular measuring instrument, through committee deliberations, item writing, pretesting, statistical item analysis, test assembly, review, and the development of interpretive manuals, can all come to nought if poor or inadequate interpretation of test scores occurs at the end of the chain. It is vitally important that teachers who use standardized tests study the interpretive manuals, know what the tests contain, and understand the possibilities and limitations of the test scores.

The development of a mathematics test is a long and expensive process, and great responsibility rests with those who produce school tests. For this reason, the Educational Testing Service is very receptive to advice and counsel from teachers everywhere in order that measuring needs may be met better. While criticism cannot change a test in print, it can certainly influence a future form or revision of the test.

The enrollments in mathematics vary according to the geographic location of the schools. In New England, the enrollments in ninth-grade general mathematics are equal to 30.8 per cent of all ninth-grade pupils, while in the South Atlantic region, the percentage is 61.—*Pamphlet No. 118, U.S. Department of Health, Education, and Welfare.*

"All practical teachers know that education is a patient process of the mastery of details, minute by minute, hour by hour, day by day. There is no royal road to learning through an airy path of brilliant generalizations. There is a proverb about the difficulty of seeing the wood because of the trees. That difficulty is exactly the point which I am enforcing. The problem of education is to make the pupil see the wood by means of the trees."—*Alfred North Whitehead.*

• TIPS FOR BEGINNERS

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia, and Joseph N. Payne, University of Michigan, Ann Arbor, Michigan

An aid to writing deductive proofs in plane geometry

by John F. Schacht, Bexley High School, Columbus, Ohio

In a recent article in THE MATHEMATICS TEACHER, Rosskopf and Exner suggest that " . . . teachers and textbooks should do a better job of introducing high school students to the concepts of logic."[1] The manner in which the authors present the meanings of mathematical operations and concepts relating to deductive proof provides an excellent background for teachers. But just how this material can be adapted to the classroom in which relatively few students are bright enough to understand and appreciate it, is quite a problem. The basic concepts of deduction should be presented to beginners as simply and clearly as possible.

Since deductive reasoning is a method of using sets of consistent conditions together with certain definitions, assumptions, and theorems to get the conclusions implied by them, the students must learn that a deductive proof has a definite form or structure. It consists of one or more syllogisms, each of which has a major premise, a minor premise, and a conclusion. If the structure of the syllogism is correct, then the conclusion is valid. If the premises are true and if the conclusion is valid, then the conclusion is necessarily true. The sentence form of a premise can be either categorical, which can be symbolized by "All p is q," or conditional, which can be symbolized by "If p, then q." A categorical statement such as "All equilateral triangles are equiangular" can be changed to the statement "If a triangle is equilateral, then it is equiangular." For most beginning students the if-then form of the major premise is the preferred one because it seems to be easier for them to identify the given conditions and the implied conclusion.

Since the syllogism is the basic unit of a proof, its structure is highly important. The major premise is a definition, an assumption, or a theorem. Stated either in the form "If p, then q," or "All p is q," the major premise means that all of the elements of the class of p's are wholly included in the class of q's. This can be illustrated by the diagram in Figure 1, shown on the following page.

The minor premise asserts that some particular element, such as x, is a member of the class of p's. The conclusion is the

[1] Myron F. Rosskopf and Robert M. Exner, "Some Concepts of Logic and Their Applications in Elementary Mathematics," THE MATHEMATICS TEACHER, XLVII (May 1955), 290.

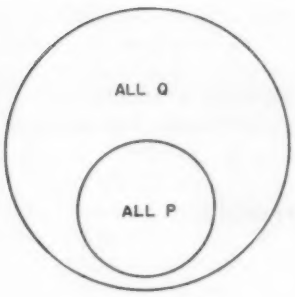


Figure 1

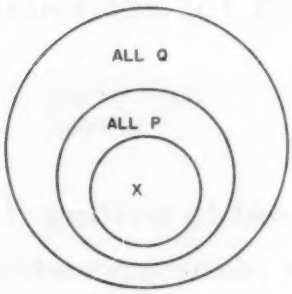


Figure 2

assertion that x is necessarily a member of the class of q's. The diagram in Figure 2 illustrates the complete syllogism.

Although most students of geometry can readily learn the structure of a simple syllogism, they experience considerable difficulty in understanding or writing a proof that involves several syllogisms.

This is particularly true when a minor premise involves the combination of two or more previously deduced conclusions. Some of the difficulty is also due to the fact that in the traditional two-column form of a proof the major premise appears as a reason or authority to the right of the corresponding conclusion. Many beginning students do not realize that there must be a logical connection between the if-clause of the authority and the minor premise, and also between the then-clause of the authority and conclusion. Moreover, they frequently include statements which are irrelevant to the ultimate conclusion, and they often write statements which are not in logical sequence.

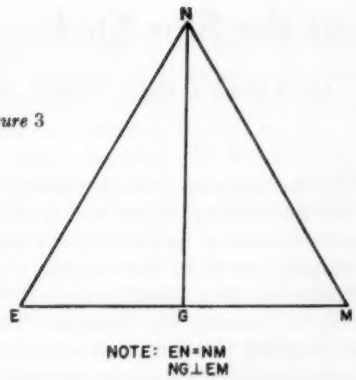
For many years the writer has been using a device consisting of linkage lines which seems to eliminate some of the difficulties and clarify the process. This device is illustrated on the facing page.

Notice how the line joining Statement 1, the minor premise, and the if-clause of Authority 2, the major premise, emphasizes that the particular element ($EN = MN$ in ΔEMN) is a member of the class of all triangles with two equal sides. Stating it another way, the line shows how the facts in Statement 1 fulfill the conditions of the major premise in Authority 2. Notice also the relation between the conclusion in Statement 2 and the then-clause of Authority 2, which completes the first syllogism in the proof. Likewise, the line joining the minor premise in Statement 3 to the if-clause of the major premise in Authority 4 shows that the conditions for the major premise are satisfied, leading to the conclusion in Statement 4. Hence, Statements 3 and 4 with Authority 4 constitute the second syllogism. The conclusion of the third syllogism is contained in Statement 5 using the facts from Statements 1, 2, and 4 as the minor premise and Authority 5 as the major premise. The fourth and final syllogism leads to the desired conclusion in Statement 6.

The lines to the left of the statements help identify the individual syllogisms in the entire proof, and they show the sequential arrangement of the four syllogisms. After several practice exercises most students need only those lines on the left of the statements, and it is usually not long until they, too, can be eliminated.

Illustration:

Figure 3



Given: ΔEMN in which $EN = MN$, and G is a point on EM such that GN is the median to the side EM.
Prove: $\angle ENG = \angle GNM$

Proof

Statements	*Authorities*
1. $EN = MN$ in ΔEMN	1. Given
	2. If two sides of a triangle are equal,
2. $\therefore \angle E = \angle M$.	the angles opposite them are equal.
3. GN is the median to the side EM.	3. Given
	4. If a line segment is a median to a side of a triangle,
4. $\therefore EG = GM$.	it divides that side into two equal parts.
Statements 1, 2, & 4	5. If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of another,
5. $\therefore \Delta EGN = \Delta GMN$	the triangles are congruent.
	6. If two triangles are congruent,
6. $\therefore \angle ENG = \angle GNM$	the corresponding angles are equal.

Meet the Slip Sticks

by William J. Giffel, Isaac C. Elston Senior High School, Michigan City, Indiana

Why not organize a mathematics club in your high school? If you feel it cannot hold the interest of your students, a pleasant surprise can be in store for you.

When organizing a mathematics club, it is well to consider the question, "Why does a student join a mathematics club?" Most students probably have two reasons. First, they have an interest in mathematics. They enjoy working with it and are seeking further contact with the subject. Secondly, they join to help satisfy a desire for social experience.

Perhaps it would help to acquaint you with the "Slip Sticks" of the Isaac C. Elston Senior High School. Their activities and projects may aid you in developing a successful mathematics club program in your school. It seems necessary at this point to mention that the extracurricular program in our high school is conducted outside regular school hours, either at noon or after school. Students are encouraged but never forced to participate. Because many of our students ride buses after school, the noon hour was chosen for the weekly meeting time.

The club had its beginning five years ago when the sponsor moved into teaching senior high school mathematics. At that time an interest in the slide rule was indicated by several students; so a club was organized, a constitution written, and a name, "Slip Sticks," chosen. The slide rule proved unsuccessful as a club project for it required too much sponsor participation and more outside practice and effort by the members than they were willing to give. The club-project idea was abandoned and, to date, not revived.

The Slip Sticks program now tries to give each member an opportunity to investigate a subject in which he is interested and to report his findings to the group. When the student has chosen his topic and made some preliminary research, he selects, a month or two in advance, the date upon which he will report.

The subjects of reports vary widely. Biographies of famous mathematicians, histories of numbers and mathematics, puzzle problems, a demonstration of elementary calculus, astronomy, geometric proofs and constructions, and an introduction to logarithms are but a few of the topics reported on. The selection of topics has been greatly facilitated by the mathematical materials on hand. By carefully selecting a few books each year to be added to the mathematics library, the results have been profitable and rewarding. A high school student's knowledge of mathematics is naturally limited, but provided with ample materials through which to browse, he will find many topics of interest to him. Reference to the March 1954 issue of THE MATHEMATICS TEACHER will provide one with an extensive listing of worthwhile mathematics books.

Out-of-school activities of the Slip Sticks are carried on to further the interest in mathematics and to promote social adjustment. Last year the group sponsored the following: a visit to the local Purdue University Extension; a Saturday trip to the Museum of Science and Industry in Chicago; and a beach party at the close of school. On the agenda for this year are a Christmas party at a member's home and a trip to "Cinerama" in Chicago. A visit to a local gauge and instruments factory is contemplated in the spring, and the annual beach party will be held in June.

Last fall, 117 students from Michigan City attended the Sci-Math Assembly at Purdue University. Each year the Slip Sticks, in cooperation with the mathematics classes, sponsor a bus trip to the

Assembly, where Indiana high school students have the opportunity to display projects of a mathematical or scientific nature. Our student-made projects dealing with geometry were displayed, and on two past occasions, members of the club talked about mathematics on the assembly program. This trip is an important occasion to the members, and considerable enthusiasm is aroused.

The possibility of making another 8 mm movie dealing with mathematics has been discussed. Last year's film, "How to Make a Window Transparency," was crude but was considered a satisfactory first try. Making another film this year has been considered, but at present the idea has reached only the discussion stage.

To gain membership in the Slip Sticks, a prospective member must submit a 350-word paper on some phase of mathematics within four weeks after he starts attending the meetings. A committee of four members receives, reads, grades, and reports its findings to the club. The average grade assigned each paper is recorded in the minutes of the club. The constitutional amendment, voted by the club members, requiring the 350-word paper for membership is a recent innovation, but it has provided the club with members who have a genuine interest in mathematics. To retain membership, he must attend regularly and show an interest in the activities by actively participating.

As the sponsor of the Slip Sticks, I enjoy an interesting and rewarding experience. I find the members eager to learn and willing to assume responsibility. Why not organize a mathematics club in your school? When you do, be prepared for the pleasure of observing the members gain greater interest and knowledge in mathematics. Your time spent in organization will provide many hours of satisfaction in the future.

Let's credit all of the 1900 A.D. high school pupils with having had work in mathematics. Since only eight per cent of all youths of high school age were in the public high schools at all in 1900, it follows that no more than eight per cent of all American youths could possibly have been taking any work in public high school mathematics in the "good old days" at the turn of the century.

In 1950, sixty-four per cent of all youngsters of high school age were attending the public high school (another thirteen per cent were in private schools). If we make allowances for repeaters, not 98 per cent but about 95 per cent of the ninth-graders were enrolled in high school mathematics in 1950. This figures out at about 60 per cent of all youngsters of high school age in this country, as compared with the figure of eight per cent for 1900. The ratio here is about 7½-to-1 in favor of the 1950 public high schools.
—*Taken from* Midland Schools, *May 1957, "On Hoodwinking the Public," by Harold C. Hand.*

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D.C.

State representatives of The National Council of Teachers of Mathematics

Below is the list of state representatives of the Council for the school year 1957–58. The activity of the state representatives, working with the Committee on Membership, was an important factor in our increase of 2400 in number of members and subscribers during the 1956–57 school year. The indications at this point of the present school year are that we shall have another increase equal to, or greater than, this. We are much indebted to the state representatives for their support.

Margaret M. Holland
Route 3, Box 728
Birmingham 8, Alabama

M. Willene Neely
Tucson High School
Tucson, Arizona

Charles Pitner
Harding College
Searcy, Arkansas

Joseph L. Slack
Chico State College
Chico, California

Paul Klipfel
1004 Las Raposas Road
San Rafael, California

John S. Herman
1001 Oxford Court
Bakersfield, California

William H. Glenn
500 Arroyo Drive
South Pasadena, California

Edwin Eagle
5039 Campanile
San Diego 15, California

Gunhild Gustafson
337 Belford Avenue
Grand Junction, Colorado

Kenneth Fuller
Teachers College
New Britain, Connecticut

Russell Dineen
1804 North Monroe Street
Wilmington, Delaware

Veryl Schult
Sheraton-Park Hotel
Washington 8, D.C.

Ethel Harris Grubbs
751 Fairmont Street, N. W.
Washington 1, D.C.

Robert Kalin
School of Education
Florida State University
Tallahassee, Florida

Kenneth P. Kidd
School of Education
University of Florida
Gainesville, Florida

Kathleen Dolvin
1015 North Main Street
College Park, Georgia

Raymond Hegg
Moscow High School
Moscow, Idaho

LeRoy Sachs
Cahokia Commonsfields
High School
Range and Jerome Lane
East St. Louis, Illinois

Philip Peak
School of Education
Indiana University
Bloomington, Indiana

E. Glenadine Gibb
Iowa State Teachers College
Cedar Falls, Iowa

Gilbert Ulmer
University of Kansas
Lawrence, Kansas

Bernice Wright
Western Kentucky State
Teachers College
Bowling Green, Kentucky

Lurnice Begnaud
P. O. Box 557
Lafayette, Louisiana

Houston T. Karnes
Louisiana State University
College of Arts & Sciences
Baton Rouge 3, Louisiana

Wilma Rollins
13 Emerson Street
Sanford, Maine

Herbert R. Smith
Baltimore Polytechnic Institute
North Avenue and Calvert
Street
Baltimore 2, Maryland

S. Leroy Taylor
Administration Building
3 East 25th Street
Baltimore 18, Maryland

Janet S. Height
41 Richardson Avenue
Wakefield, Massachusetts

Mary E. Reed
1070 McAllister
Benton Harbor, Michigan

Ardith Erickson
St. Louis Park Junior High School
St. Louis Park, Minnesota

Eleanor Walters
Delta State College
Cleveland, Mississippi

Lee O. Jones
William Jewell College
Liberty, Missouri

Adrien L. Hess
Montana State College
Bozeman, Montana

Virginia Lee Pratt
Central High School
Omaha 2, Nebraska

Vera Z. Warren
528 Reno Avenue
Reno, Nevada

H. Gray Funkhouser
26 Elliot Street
Exeter, New Hampshire

Hubert B. Risinger
18 Summit Street
East Orange, New Jersey

Arthur A. LePori
East Orange High School
East Orange, New Jersey

Reba Jinkins
1120 Pile
Clovis, New Mexico

Lucile Brooks
Marcellus Central School
Marcellus, New York

Harry D. Ruderman
2624 Davidson Avenue
Bronx 68, New York

Miss Clyde Hunter
900 West Vance Street
Wilson, North Carolina

Ernest M. Pletan
1517 North Third Street
Grand Forks, North Dakota

Frances Theaker
679 Manchester Road
Mansfield, Ohio

Eunice Lewis
University High School
Norman, Oklahoma

Eva Burkhalter
Klamath Union High School
Klamath Falls, Oregon

H. L. Krall
Pennsylvania State University
University Park, Pennsylvania

Earle F. Myers
University of Pittsburgh
Pittsburgh 13, Pennsylvania

M. Albert Linton
William Penn Charter School
Philadelphia, Pennsylvania

M. L. Herman
Moses Brown School
Providence 6, Rhode Island

Gwen Broman
Groton, South Dakota

Jane Wright
4201 Colorado Avenue
Nashville, Tennessee

Ella S. Porter
5655 San Felipe
Houston 19, Texas

Eva A. Crangle
241 North Second, West
Salt Lake City 16, Utah

John G. Bowker
Middlebury College
Middlebury, Vermont

Emilie Holladay
93—33rd Street, Apt. 2
Newport News, Virginia

R. W. James
Armstrong High School
North 31st Street
Richmond 23, Virginia

Harold J. Hunt
4504 Woodlawn Avenue
Seattle 3, Washington

Fred Kramlich
Lewis and Clark High School
Spokane, Washington

Kathryn W. Lynch
923 Sixth Avenue
St. Albans, West Virginia

Miss Elli Otteson
705 Whipple
Eau Claire, Wisconsin

Lydia R. Goerz
2204—60th Street
Kenosha, Wisconsin

Frank C. Denney
3628 Dunn Avenue
Cheyenne, Wyoming

Leonard Lucas
62 Jackman Road
Bowmanville, Ontario, Canada

Report of the Committee on Supplementary Publications

by Henry Swain, New Trier Township High School, Winnetka, Illinois

The publications of the National Council of Teachers of Mathematics fall into three categories: (1) the Journals, including THE MATHEMATICS TEACHER, *The Arithmetic Teacher*, and *The Mathematics Student Journal;* (2) the Yearbooks; and (3) all other publications of the Council. The last of these categories comes under

the supervision of the Committee on Supplementary Publications. General supervision and policy making for all the publications of the Council are handled by the Publications Board.

Many members of the Council are familiar with the pamphlets serving mathematics teachers and students which have been published in the last four or five years. These include *Geometry Growing, Recreational Mathematics, How to Study Mathematics, Byroads of Algebra, A Portrait of 2, Designing the Mathematics Classroom,* and *Paper Folding for the Mathematics Class.* A series of pamphlets called the "How-to Series" come in $8\frac{1}{2} \times 11''$ size, punched for notebook use. Pamphlets published in this series are *How to Use Your Bulletin Board, How to Develop a Teaching Guide in Mathematics, How to Use Field Trips in Mathematics,* and *How to Use Films and Filmstrips in Mathematics Classes.* Other pamphlets that are now being published are *Mathematics Clubs for High Schools, Program Provisions for the Mathematically Gifted Student in the Secondary School, Education in Mathematics for the Slow Learner,* and *How to Use Your Library in Mathematics.* Several additional manuscripts are in various stages of preparation.

The responsibility of the Committee on Supplementary Publications is to receive manuscripts, to determine with the help of the Publications Board whether they are appropriate for NCTM publication, and to work with the authors in helping to refine the manuscripts until they are ready for publication. The actual publication from then on is supervised by the Executive Secretary, Mr. M. H. Ahrendt.

The committee also helps find authors or editors for pamphlets for which a need has been indicated.

To have the Committee on Supplementary Publications serve the profession most effectively, we must know what kind of material mathematics teachers need most, and we must be in contact with people who are willing and able to help write manuscripts.

There are three ways in which you, the reader, can help us.

1. If you have a manuscript or feel the urge to write a manuscript which you think would be of value to mathematics teachers, please send it to us. We welcome unsolicited manuscripts and give them very careful consideration.

2. If you know of someone who has written or who should be encouraged to write material appropriate for C.S.P pamphlets, please give us his name.

3. If you feel the need for a pamphlet on a particular topic, please let us know. It would help if you could name people who might be able to write on the topic you are suggesting.

If you can help in any of these three ways, write to the Chairman of the Committee on Supplementary Publications, c/o National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

We cannot finish a report like this without paying high tribute to the many people—authors, editors, committee workers—who give freely of their time and talent to make this good work possible. It is through such devoted service to the cause of improving the teaching of mathematics that the NCTM has become the organization it is.

HENRY SWAIN, *Chairman*
MILDRED KIEFFER
DWAIN SMALL
MARGARET JOSEPH
EDWIN EAGLE
J. HOUSTON BANKS
MARGUERITE BRYDEGAARD
KENNETH KIDD
JESSE OSBORNE
LAWRENCE A. RINGENBERGER
ROBERT SEBER
JAMES ULRICH

Registration reports on two recent meetings

Below are the official registration reports for two recent meetings of the Council. The meeting in Philadelphia on July 1, 1957, was of special significance because it was our first joint meeting with the National Science Teachers Association. The program, planned by the presidents of NCTM and NSTA, featured a group of outstanding leaders in both science and mathematics education.

The meeting in Northfield, Minnesota, August 18–21, 1957, was of special interest because it was the third largest Summer Meeting in the history of the Council.

Registrations at Joint Meeting of NCTM and NSTA with NEA

Philadelphia, Pennsylvania, July 1, 1957

Alabama	3	Minnesota	5
Arizona	2	Mississippi	3
Arkansas	2	Montana	4
California	8	Nebraska	2
Colorado	1	North Carolina	4
Connecticut	3	North Dakota	1
Delaware	6	New Jersey	28
District of Columbia	3	New York	13
Florida	2	Ohio	3
Georgia	2	Oklahoma	1
Idaho	1	Pennsylvania	92
Illinois	13	Tennessee	6
Indiana	4	Texas	3
Iowa	1	Virginia	10
Kansas	2	Washington	5
Kentucky	3	West Virginia	2
Louisiana	1	Wisconsin	2
Maryland	13	Foreign	5
Massachusetts	3		
Michigan	9	Total	271

Registrations at the Seventeenth Summer Meeting

The National Council of Teachers of Mathematics, Carleton College, Northfield, Minnesota, August 18–21, 1957

Alabama	1	New Jersey	7
Arkansas	3	New Mexico	1
California	7	New York	11
Colorado	7	North Carolina	1
Connecticut	3	North Dakota	4
District of Columbia	2	Ohio	12
Florida	2	Oklahoma	1
Illinois	52	Oregon	1
Indiana	28	Pennsylvania	5
Iowa	38	Rhode Island	1
Kansas	4	South Carolina	1
Kentucky	5	South Dakota	8
Louisiana	3	Tennessee	3
Maryland	1	Texas	9
Michigan	22	Washington	1
Minnesota	152	Wisconsin	30
Missouri	9	Canada	8
Montana	3	Foreign	1
Nebraska	2		
New Hampshire	2	Total	451

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D.C.

NCTM convention dates

ANNUAL MEETING

April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Hyman, Board of Education, Cleveland, Ohio

JOINT MEETING WITH NEA

June 30, 1958
Cleveland, Ohio
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

EIGHTEENTH SUMMER MEETING

August 19–20, 1958
Colorado State College of Education, Greeley, Colorado
Forest N. Fisch, Colorado State College of Education, Greeley, Colorado

CHRISTMAS MEETING

December 29–30, 1958
Sheraton-McAlpin Hotel, New York, New York
Abraham S. Kadish, High School of Fashion Industries, 225 West 24th Street, New York 11, New York

Other professional dates

Luncheon Meeting of Florida Council of Teachers of Mathematics
April 25, 1958 (Noon)
Roney Plaza Hotel, Miami Beach, Florida
Florence Shaffer, 631 N.W. 138th Street, Miami, Florida

Eighth Annual Meeting of the Association of Mathematics Teachers of New York State
May 2–3, 1958
Hotel Syracuse, Syracuse, New York
Martha Neighbour, Ithaca High School, Ithaca, New York

Spring Meeting of the Arkansas Council of Teachers of Mathematics
May 3, 1958
Lafayette Hotel, Little Rock, Arkansas
Christene Poindexter, Central High School, Little Rock, Arkansas

Tenth Annual Institute of Association of Teachers of Mathematics in New England
August 19–26, 1958
Brandeis University, Waltham, Massachusetts
Thomas F. McCarthy, Mathematics Department, Peabody High School, Peabody, Massachusetts

Annual Meeting of the Illinois Council of Teachers of Mathematics
October 4, 1958
Aurora, Illinois
Francis R. Brown, Illinois State Normal University, Normal, Illinois

The National Education Association estimates that 600,000 *new teaching positions* will be required to meet the needs of increased enrollments during the next ten years.

Of local interest

Michigan Council of Teachers of Mathematics: The Ninth Annual Conference of the Michigan Council of Teachers of Mathematics will be held at the M.E.A. Camp, St. Mary's Lake, Battle Creek, Michigan, on May 2–3–4, 1958. Registration will start at noon on Friday, May 2, and the conference will close with dinner Sunday noon, May 4. The theme of the conference will be "Insights into Modern Mathematics."

In addition to the general sessions, seminar-type group meetings will be held in which topics of "modern" mathematics will be discussed. There will be discussion groups dealing with classroom problems. These are planned for elementary school teachers as well as for junior high and senior high school teachers.

The camp is pleasantly located on St. Mary's Lake, four miles north of Battle Creek. Recreational facilities are provided, meals are served in the main dining hall, and dormitory-type sleeping accommodations are available for about 150 persons. Hotel accommodations are available in nearby Battle Creek for those who prefer them. The charge for the conference (including all accommodations) is modest, and members of the Michigan Education Association are entitled to a substantial discount. To be assured of sleeping accommodations at the camp, reservations should be made in advance. Inquiries and request for reservations may be addressed to Miss Carolyn Ingham, 500 East William Street, Ann Arbor, Michigan.

Galileo Galilei (1564-1642), leader in experimental physics and astronomy, upon leaving his post at the telescope one night, was heard to remark

"but first I took First and Second Year Algebra"

There's an anachronism somewhere, but he *has* got a point. For **First Year Algebra** and **Second Year Algebra,** by *Hart, Schult,* and *Swain,* provide a solid foundation in algebraic principles. You'll like the

- *sound approach . . . excellent organization*
- *beautiful and functional format*
- *tested instructional methods*
- *many carefully graded problems and exercises*

Answers • Teachers Manuals • Keys

D. C. HEATH AND COMPANY

Sales Offices: Englewood, N.J. Chicago 16 San Francisco 5 Atlanta 3 Dallas 1 *Home Office:* Boston 16

Please mention THE MATHEMATICS TEACHER when answering advertisements

WILEY BOOKS

A unified treatment of the principles of essential business mathematics

MATHEMATICS IN BUSINESS

By Lloyd L. Lowenstein
Arizona State College

Prepares students for further courses

This text is designed to give a foundation in business mathematics built upon *understanding* rather than upon mere formulas. Treating the subject clearly, logically, and without unnecessary digressions, it gives students the background they will need for further courses in the mathematics of finance, accounting, business statistics, insurance, and other business subjects. Special care has been taken to present the material in a unified fashion. All definitions are as precise as possible, and new ideas are introduced one at a time. There is frequent reference to, and application of, principles discussed earlier in the book.

Features varied problems

An outstanding feature is the large collection of varied problems. These are carefully graded and arranged in the "twin problem" manner—an odd numbered problem and its immediate even-numbered successor are of the same type and degree of difficulty. Answers to the odd-numbered problems are provided in the text. Time-value diagrams for interest and discount problems and related problems are introduced early and used throughout the text.

Encourages development of new ideas

Mathematics in Business leads students to develop new ideas for themselves. Frequently problems are presented which involve ideas not studied in detail until the following section or chapter. If they have fully understood the principles studied before these problems, they will have no difficulty with them and will at the same time discover new problems for themselves. For those students unable to develop new ideas for themselves, the development soon appears in the text.

1958 **364 pages** **$4.95**

Send today for an examination copy

JOHN WILEY & SONS, Inc.
440 Fourth Avenue New York 16, N.Y.

Please mention THE MATHEMATICS TEACHER when answering advertisements

FISK Teacher's Agency

28 E. Jackson Blvd. - Chicago 4, Ill.

We make recommendations for positions only to administrators who request us to do so. The most important part of our work is to find the right person for the right position.

Our service is nation-wide.

Are you building? remodeling? equipping?

You Need

DESIGNING THE MATHEMATICS CLASSROOM

by LAWRENCE P. BARTNICK

•

Among the many topics discussed are these: lighting, chalkboards, storage, display and exhibit facilities, furniture, mathematical equipment, tools, supplies, enrichment materials, pictures. Gives sources of supply and prices.

Contains a good group of typical floor plans for the mathematics classroom and laboratory.

44 pages Price $1.00

•

Postpaid if you send remittance with order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1201 Sixteenth Street, N.W., Washington 6, D.C.

MATHEMATICAL TEACHING MODELS

For every course where geometry is used, taught or applied, there is a model available for use. Theorems can be illustrated in seconds, and the impact of an exact picture remains fixed in the student's mind. These models are suitable for courses in:

- PLANE GEOMETRY
- SOLID GEOMETRY
- DESCRIPTIVE GEOMETRY
- MECHANICS
- MECHANICAL DRAWING
- INDUSTRIAL DESIGN

THEY HAVE THESE OUTSTANDING ADVANTAGES:

- CLEAR PLASTIC
- LIGHT WEIGHT
- CONVENIENT SIZE
- MODERATE COST
- SINGLE UNITS OR SETS

The clear plastic models permit the instructor to show the student where diagonals intersect INSIDE the bodies. The moving models in mechanics show the student what happens when gears mesh; the block type models permit the building of figures; the sets show the sections made by the intersections of cones and other solid figures by planes and by each other. Hundreds of models are available for almost every conceivable application.

MODEL 132: Plane section of a right circular cylinder (ellipse).

These models are products of West Germany and are presented by Physics Research Laboratories, Inc. which is known for the fine European made instruments it brings to American laboratories.

WRITE FOR COMPLETE CATALOG

Physics Research LABORATORIES *House of Fine Instruments*

CONSULTANTS IMPORTERS P.O. Box #555 Hempstead, L. I., N. Y.

THE MATHEMATICAL DETECTIVE

by Florence Bourgeois, M.A.
Springside School

Introducing symbols, the equation, and featuring

$$F_1 \times F_2 = P$$

Illustrated by some formulas and containing up-to-date problems.

Children's reading . . . a pamphlet illustrated by Helen-Lee Jones.

Single copies $1.00 each
By the dozen $.80 each

Discount on quantity orders

Write: Box 4338
Philadelphia 18, Pennsylvania

MATHEMATICS CONTEST
Information sent on request

STUDENT SLIDE RULES

FIELD WORK IN MATHEMATICS
By Shuster and Bedford
Other special books to enrich mathematics teaching

FIELD WORK INSTRUMENTS
Transits, Plane Tables, Alidades, Sextants, Hypsometers, Angle Mirrors, Tapes, Leveling Rods, Ranging Poles

GROVE'S MOTO-MATH SET
To illustrate all plane figures in Modern, High School and College Mathematics

MULTI-MODEL GEOMETRIC CONSTRUCTION SET
For Solid Geometry and Modern Mathematics

VISUAL AIDS AND MODELS
Send for Literature and Prices

YODER INSTRUMENTS
The Mathematics House Since 1930
East Palestine, Ohio

BOOKS
from Prentice-Hall...

1 Essential Mathematics for College Students

by FRANCIS J. MUELLER,
Maryland State Teachers College

This new text contains a thorough and meaningful review of basic arithmetic processes. Highly conducive to self-instruction or independent work, the text requires no prerequisites. It is so organized that when the student has completed and detached all exercise pages, he still has the complete text available for future reference.

Approx. 288 pp. • Paper Bound • Pub. 1957
Text price $3.95

2 Arithmetic: Its Structure and Concepts

by FRANCIS J. MUELLER,
Maryland State Teachers College

Written at a mature level this text views arithmetic as a guided system of thought based on reason and understanding rather than the application of mechanical rules.

Professor Mueller presents the material in an original light, dealing with arithmetic concepts in accordance with what he expresses as "the three basic things we do with numbers"—
(1) Synthesis (2) Analysis (3) Comparison.
This breakdown serves as a unifying frame of reference for his discussions.

279 pages • Published 1956 • Text price $6.00

3 Introduction to Finite Mathematics

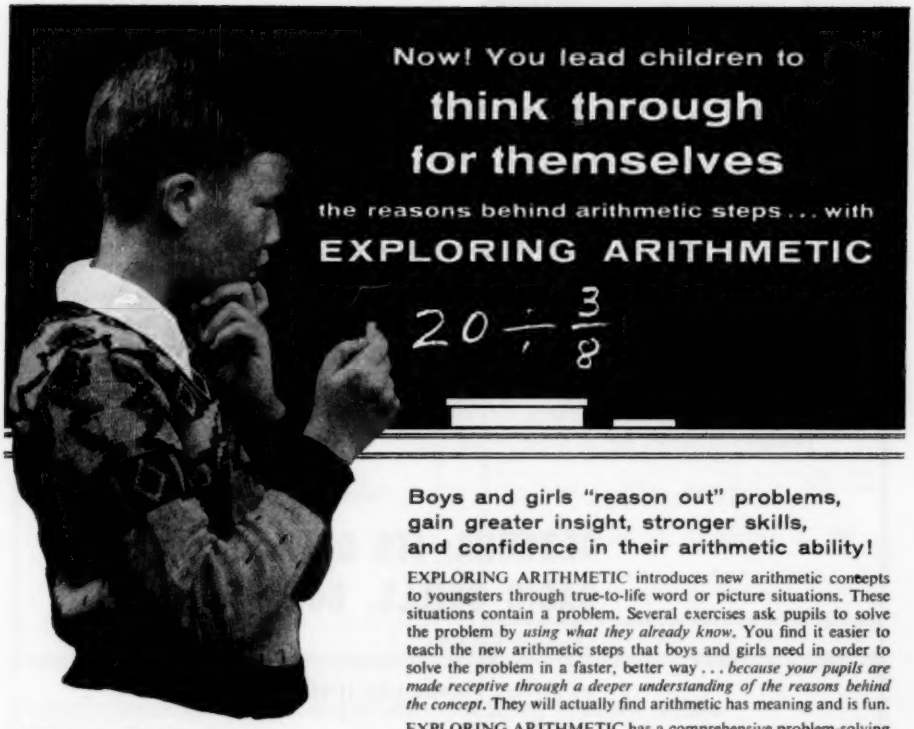
by JOHN G. KEMENY, J. LAURIE SNELL and GERALD L. THOMPSON, all of Dartmouth College

This unique text is designed to introduce concepts in modern mathematics early in the student's career, and provide the foundation for important topics in mathematics outside the calculus. Aimed at the freshman level, it includes an introduction to: logic, set theory, partitions, probability theory, matrix theory, mathematical models, game theory and linear programming. This course has been successfully tested.

372 pp. • Pub. 1957 • Text price $5.50

To receive approval copies promptly, write: BOX 903

PRENTICE-HALL, Inc.
Englewood Cliffs, New Jersey



Boys and girls "reason out" problems, gain greater insight, stronger skills, and confidence in their arithmetic ability!

EXPLORING ARITHMETIC introduces new arithmetic concepts to youngsters through true-to-life word or picture situations. These situations contain a problem. Several exercises ask pupils to solve the problem by *using what they already know*. You find it easier to teach the new arithmetic steps that boys and girls need in order to solve the problem in a faster, better way . . . *because your pupils are made receptive through a deeper understanding of the reasons behind the concept*. They will actually find arithmetic has meaning and is fun.

EXPLORING ARITHMETIC has a comprehensive problem-solving program, with an abundance of word problems and practice exercises to nail down the stronger understanding youngsters acquire. There is a spiral reteaching program, thorough testing, plus good organization, an orderly format and the writing is simple and clear. This program has the most fascinating illustrations ever used in arithmetic textbooks. Write for an illustrated brochure or ask your Webster Representative for examination copies. See for yourself why this new program is so remarkable.

Also see these other Webster publications:

THE NEW GOALS IN SPELLING,
features phonics and word-structure; grades 2-8

THE NEW WEBSTER LANGUAGE,
teaches language skills step by step; grades 3-6

WEBSTER CLASSROOM SCIENCE LIBRARY,
12 science books for reading and activities; grades 4-8

EXPLORING ARITHMETIC
by Jesse Osborn, Adeline Riefling and Herbert F. Spitzer.

Grades 3-8; Complete and functional Teachers' Editions for each grade.

WEBSTER PUBLISHING CO.
1808 Washington Avenue • St. Louis 3, Missouri

EXPLORING ARITHMETIC uses many special visual teaching features, such as the Number Line and the Abacus. Use this coupon to order your FREE Webster Number Line:

Webster Publishing Company
Please send me a Webster Number Line:

Name_____

School_____

Address_____

City_____ Zone_____

State_____

A complete, effective mathematics program—

USING MATHEMATICS 7-8-9

by Henderson and Pingry

An interesting, challenging series for 7th, 8th, and 9th grades, with an easy reading level. Problems interest boys and girls, farm and city pupils. Color drawings and cartoons provide motivation and illustrate principles. Self-teaching methods help pupils to discover principles for themselves, making clear the reason *why* as well as the *how*. Test Problems Workbooks (7-8-9), Teacher's Editions (7-8) and Teacher's Manuals for all three books.

ALGEBRA: ITS BIG IDEAS AND BASIC SKILLS, Books I and II,

Second Edition

by Aiken, Henderson, and Pingry

Organized around the big ideas and basic skills of algebra for better understanding. Many problems to fix and apply skills; optional topics and problems for superior students. Two colors throughout for effective learning. 160 new drawings add a visual dimension to ideas. Exercises help students to discover principles. Self tests check progress. Tests and Teacher's Key. Complete Teacher's Key for Book I.

PRINCIPLES OF MATHEMATICS

by Allendoerfer and Oakley

For the 12th Grade advanced mathematics course. Combines advanced algebra, trigonometry, analytic geometry, and calculus.

McGRAW-HILL BOOK COMPANY

New York 36 Chicago 30 Dallas 2 San Francisco 4

STRAIGHT LINE UNIT, NO. 186

This interesting kit discusses the problem of drawing an original straight line without the use of a straightedge.

Shows how the straight line problem can be solved by the use of linkages. Also explains other uses of linkages.

Contains equipment for constructing several small operating models of linkages. Instruction leaflet gives directions, with bibliography for further reading.

75¢ each. 3 for $1.50.

National Council of Teachers of Mathematics
1201 Sixteenth Street, N.W.
Washington 6, D.C.

WESLEYAN UNIVERSITY
GRADUATE SUMMER SCHOOL FOR TEACHERS

EARN THIS UNIQUE DEGREE:
Master of Arts in Liberal Studies

The M.A.L.S. degree—established by Wesleyan University in 1953—is designed especially for men and women teachers who are ambitious for personal and professional growth—teachers who want to fit themselves more fully for the new opportunities in American education. The program of 30 hours permits work in two or more related subjects. It may include introductory courses in subjects not previously studied. Courses in Education are not required. Small classes are taught by a distinguished faculty.

The emphasis is on correlative study. For example, a major in MATHEMATICS offers six or more courses in Mathematics designed to strengthen command of mathematical principles and to provide the knowledge of modern mathematics necessary to teach the revised high school curriculum recommended by the Commission on Mathematics of the College Entrance Examination Board; a background course in the growth of Western civilization; electives in science or other subjects.

The Certificate of Advanced Study is awarded upon satisfactory completion of 30 hours beyond the Master's degree. **CULTURAL, RECREATIONAL PROGRAM** includes art, music, tennis, swimming, tours.

CLASSES—JULY 2 TO AUGUST 12, 1958

Catalog: Math Major, Summer School,
Box 39 Wesleyan Station, Middletown, Conn.

Fundamental Mathematics—meeting today's needs in high schools and junior colleges

Smith and Lankford

ALGEBRA ONE
ALGEBRA TWO

"So many well-conceived ideas . . . excellent innovations."
The Clearing House

Smith and Ulrich

PLANE GEOMETRY
SOLID GEOMETRY

"Well organized . . . one of the best written books on plane geometry in recent years . . . the solid geometry is a splendid achievement."
Junior College Journal

Smith and Hanson

TRIGONOMETRY

Just published, to round out the student's mathematical training in a terminal course, or in preparation for higher mathematics.

WORLD BOOK COMPANY Yonkers-on-Hudson, New York

Factors to keep in mind when students ask about slide rules

BY ROBERT JONES, *Manager of Educational Sales, Frederick Post Company*

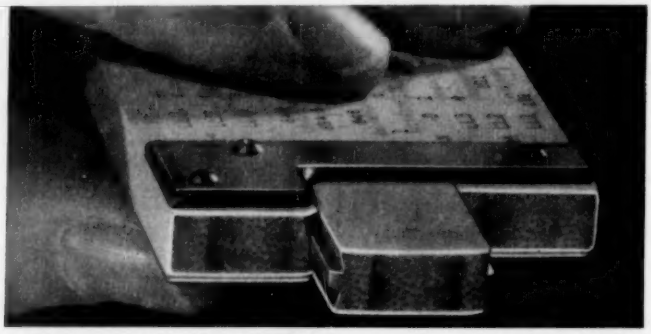


Photo of end of slide rule showing laminated bamboo construction exclusive in all POST rules.

Construction

Laminated bamboo rules (photo above) resist expansion and warping in even the most humid weather. This dimensional stability means smooth sliding and more dependable performance, all year around.

Bamboo is a unique material because it contains natural, sealed-in silica particles that provide self-lubrication. Since all POST rules are made of laminated bamboo, there is never a need for lubricants. In fact, the longer your students use a POST rule, the easier the slide is to operate.

Graduations

"Bargain" rules selling in the lower price bracket often have imprinted or stamped markings. Graduations are frequently inaccurate and soon fade or rub off.

All POST rules are "engine-divided". By this process, graduations are cut into the white celluloid face by a precision-controlled knife. The markings last as long as the rule itself. The small price difference between a "bargain" rule and a bamboo, engine-divided POST rule is more than made up in accurate, long lasting performance.

Range of Rules

For secondary school work, the widely used POST 10" Mannheim type STUDENT rule or the POST UNIVERSAL rule give greatest value. However, for the student going on to engineering school, the POST VERSALOG is recommended in his senior high school year because of its advanced scale arrangement, new end zone designations, greater accuracy and color coded trig scales.

For an examination sample of any of these POST rules, send your "on approval" order to Bob Jones, Manager of Educational Sales, Frederick Post Co., 3650 N. Avondale, Chicago 18, Ill.

Special classroom prices for 4 POST slide rules

1. **Mannheim type, 10" STUDENT RULE, No. 1447** $2.81
   (Popularly priced rule that has brought a flood of orders from secondary school teachers during the past two years.)
2. **10" UNIVERSAL RULE, No. 1452WL** ... $7.50
   (Standard Mannheim scales with inch and metric scales on edges. Extra gauge marks for faster calculating. Leather case included.)
3. **VERSALOG professional rule, No. 1460L** $19.69
   (Includes handsome leather case and cloth bound 120 page self-instruction manual)
4. **VEST POCKET RULE, No. 1444K** $3.56
   (5" scale, weighs ½ oz. Face of rule has A, B, CI, C and D scales. Reverse side has S, L and T scales. Inch scale on side. Includes leather pocket case.)

Detailed information on all rules is supplied in POST'S *Slide Rule Catalog, available on request.*

48" x 52" POST Teaching Aid Wall Chart for slide rule instruction $1.00

POST

JUST OFF THE PRESS

STUDENT-CENTERED PAMPHLETS

PROGRAM PROVISIONS FOR THE MATHEMATICALLY GIFTED STUDENT IN THE SECONDARY SCHOOL, edited by E. P. Vance, with contributions by Julius H. Hlavaty, Richard S. Pieters, and LeRoy Sachs

Discusses approaches to the development of a mathematics program for the gifted.

Reports on programs developed in a variety of types of schools.

Gives the recommendations of several committees and commissions.

32 pages **75¢ each**

EDUCATION IN MATHEMATICS FOR THE SLOW LEARNER, by Mary Potter and Virgil Mallory

Contains a comprehensive discussion of the special characteristics and problems of the slow learner, with a useful list of do's and don't's.

Gives program and curriculum suggestions, with illustrations from practice.

Provides a large bibliography of professional materials and textbooks.

36 pages **75¢ each**

MATHEMATICS CLUBS IN HIGH SCHOOL, by Walter Carnahan

An inclusive practical discussion of mathematics clubs. Discusses objectives, organization, officers, constitution, activities, programs, and related matters.

Gives many ideas and sources of material for club programs, with a report of some actual programs.

Contains bibliographies of source materials and a list of present active clubs.

32 pages **75¢ each**

HOW TO USE YOUR LIBRARY IN MATHEMATICS, by Allene Archer

No. 5 in the How-to-Do-It Series.

Discusses purposes for which the library is used, guidance in the use of the library, desirable outcomes, types of reference materials, topics, and projects.

Contains information on historical reports, things to make, great mathematicians, and famous quotations about mathematics.

6 pages **40¢ each**

PAPER FOLDING FOR THE MATHEMATICS CLASS, by Donovan A. Johnson

Gives directions for forming or illustrating by paper folding the basic constructions, geometric concepts, circle relationships, products and factors, polygons, knots, polyhedrons, symmetry, conic sections, recreations.

Illustrated with 139 drawings.

36 pages **75¢ each**

Shipped postpaid if you send remittance with order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1201 Sixteenth Street, N.W. **Washington 6, D.C.**

NEW—Welch—CATALOG

of

Mathematics Equipment and Supplies

Just off the press!

COMPLETELY REVISED

WRITE FOR YOUR COPY TODAY!

This entirely new catalog of equipment for the Mathematics Department includes

The Schacht Devices for Dynamic Geometry

Drawing Instruments and Kits

Slide-rules—Calipers

Models—Charts

Projection Equipment and Film strips

and many other items of particular interest to the Progressive Mathematics Teacher.

Serving the Schools for over 70 years.

W. M. WELCH SCIENTIFIC COMPANY

DIVISION OF W. M. WELCH MANUFACTURING COMPANY

1515 Sedgwick Street, Chicago 10, Illinois, U.S.A.

Manufacturers of Scientific Instruments and Laboratory Apparatus

Please mention THE MATHEMATICS TEACHER when answering advertisements